Algorithmic Complexity

Nelson Padua-Perez

Bill Pugh

Department of Computer Science

University of Maryland, College Park

Algorithm Efficiency

- Efficiency
  - Amount of resources used by algorithm
    - Time, space
- Measuring efficiency
  - Benchmarking
  - Asymptotic analysis

Benchmarking

- Approach
  - Pick some desired inputs
  - Actually run implementation of algorithm
  - Measure time & space needed
- Industry benchmarks
  - SPEC – CPU performance
  - MySQL – Database applications
  - WinStone – Windows PC applications
  - MediaBench – Multimedia applications
  - Linpack – Numerical scientific applications

Benchmarking

- **Advantages**
  - **Precise information for given configuration**
    - **Implementation, hardware, inputs**
- **Disadvantages**
  - **Affected by configuration**
    - **Data sets (usually too small)**
    - **Hardware**
    - **Software**
  - **Affected by special cases (biased inputs)**
  - **Does not measure intrinsic efficiency**

Asymptotic Analysis

- Approach
  - Mathematically analyze efficiency
  - Calculate time as function of input size n
    - $T \approx O[\ f(n)\ ]$
    - T is on the order of f(n)
    - "Big O" notation
- Advantages
  - Measures intrinsic efficiency
  - Dominates efficiency for large input sizes

Search Example

- **Number guessing game**
  - Pick a number between 1…n
  - Guess a number
  - Answer “correct”, “too high”, “too low”
  - Repeat guesses until correct number guessed

Linear Search Algorithm

- Algorithm
  - Guess number = 1
  - If incorrect, increment guess by 1
  - Repeat until correct

- Example
  - Given number between 1…100
  - Pick 20
  - Guess sequence = 1, 2, 3, 4 … 20
  - Required 20 guesses

Linear Search Algorithm

- **Analysis of # of guesses needed for 1…n**
  - **If number = 1, requires 1 guess**
  - **If number = n, requires n guesses**
  - **On average, needs n/2 guesses**
  - **Time = O(n) = Linear time**

Binary Search Algorithm

- Algorithm
  - Set Δ to n/4
  - Guess number = n/2
  - If too large, guess number – Δ
  - If too small, guess number + Δ
  - Reduce Δ by ½
  - Repeat until correct

Binary Search Algorithm

- Example
  - Given number between 1…100
  - Pick 20
  - Guesses =
    - 50, Δ = 25, Answer = too large, subtract Δ
    - 25, Δ = 12 , Answer = too large, subtract Δ
    - 13, Δ = 6, Answer = too small, add Δ
    - 19, Δ = 3, Answer = too small, add Δ
    - 22, Δ = 1, Answer = too large, subtract Δ
    - 21, Δ = 1, Answer = too large, subtract Δ
    - 20
  - Required 7 guesses

Binary Search Algorithm

- Analysis of # of guesses needed for 1…n
  - If number = n/2, requires 1 guess
  - If number = 1, requires $\log_2(n)$ guesses
  - If number = n, requires $\log_2(n)$ guesses
  - On average, needs $\log_2(n)$ guesses
  - Time = $O(\log_2(n))$ = Log time

Search Comparison

- For number between 1…100
  - Simple algorithm = 50 steps
  - Binary search algorithm = $\log_2(n)$ = 7 steps
- For number between 1…100,000
  - Simple algorithm = 50,000 steps
  - Binary search algorithm = $\log_2(n)$ = 17 steps
- Binary search is much more efficient!

Asymptotic Complexity

- Comparing two linear functions

Size	Running Time	
	n/2	4n+3
64	32	259
128	64	515
256	128	1027
512	256	2051

Asymptotic Complexity

- Comparing two functions
  - n/2 and 4n+3 behave similarly
  - Run time roughly doubles as input size doubles
  - Run time increases linearly with input size
- For large values of n
  - Time(2n) / Time(n) approaches exactly 2
- Both are O(n) programs

Asymptotic Complexity

- Comparing two log functions

Size	Running Time	
	$\log_2(n)$	$5 * \log_2(n) + 3$
64	6	33
128	7	38
256	8	43
512	9	48

Asymptotic Complexity

- Comparing two functions
  - $\log_2(n)$ and $5 * \log_2(n) + 3$ behave similarly
  - Run time roughly increases by constant as input size doubles
  - Run time increases logarithmically with input size
- For large values of n
  - Time(2n) – Time(n) approaches constant
  - Base of logarithm does not matter
    - Simply a multiplicative factor

      $\log_a N = (\log_b N) / (\log_b a)$
- Both are O(log(n)) programs

Asymptotic Complexity

- Comparing two quadratic functions

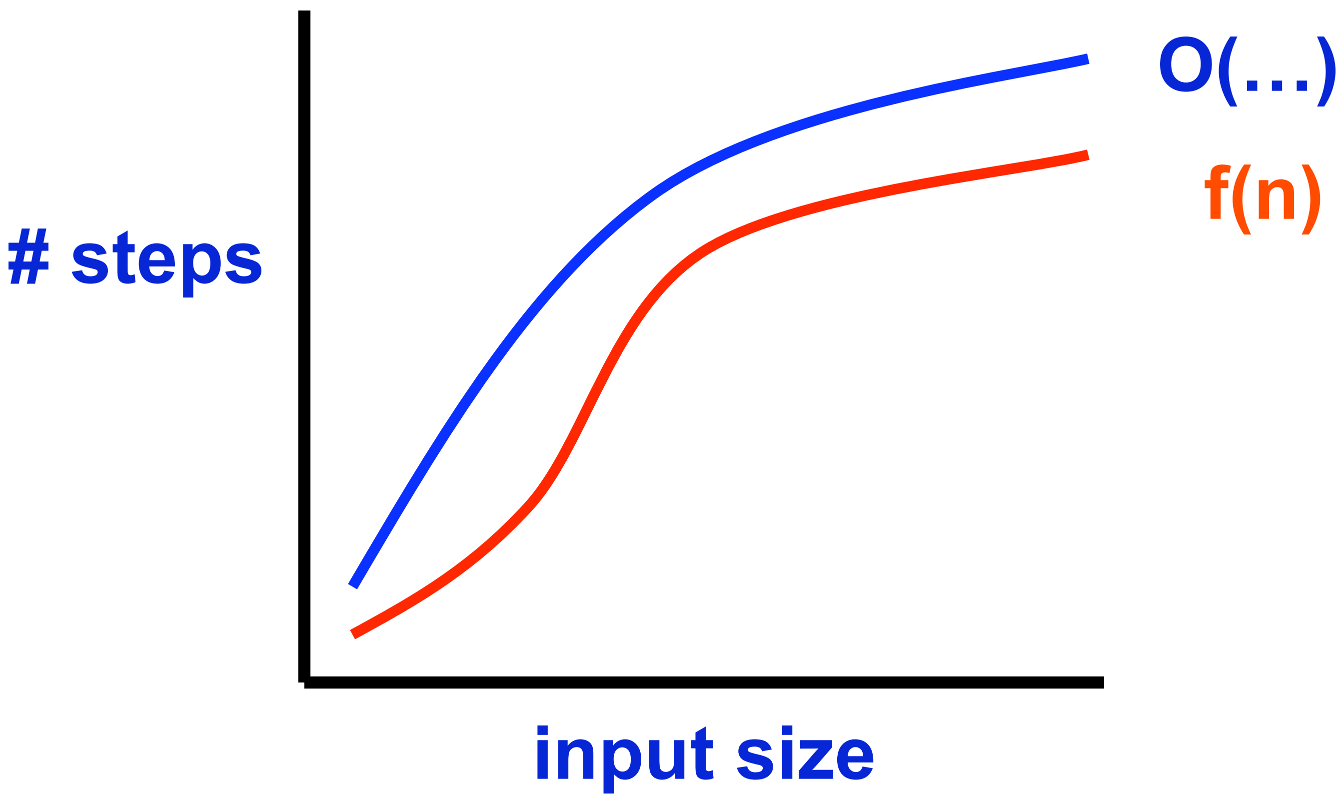
Size	Running Time	
	n^2	$2 n^2 + 8$
2	4	16
4	16	40
8	64	132
16	256	520

Asymptotic Complexity

- Comparing two functions
  - n^2 and $2\ n^2 + 8$ behave similarly
  - Run time roughly increases by 4 as input size doubles
  - Run time increases quadratically with input size
- For large values of n
  - Time(2n) / Time(n) approaches 4
- Both are O(n^2) programs

Big-O Notation

- Represents
  - Upper bound on number of steps in algorithm
  - Intrinsic efficiency of algorithm for large inputs



Formal Definition of Big-O

- Function f(n) is O(g(n)) if
  - For some positive constants M, N_0
  - $M \times g(n) \geq f(n)$, for all $n \geq N_0$
- Intuitively
  - For some coefficient M & all data sizes $\geq N_0$
    - $M \times g(n)$ is always greater than f(n)

Big-O Examples

- $5n + 1000 \Rightarrow O(n)$
  - Select $M = 6$, $N_0 = 1000$
  - For $n \geq 1000$
    - $6n \geq 5n+1000$ is always true
  - Example $\Rightarrow$ for $n = 1000$
    - $6000 \geq 5000 + 1000$

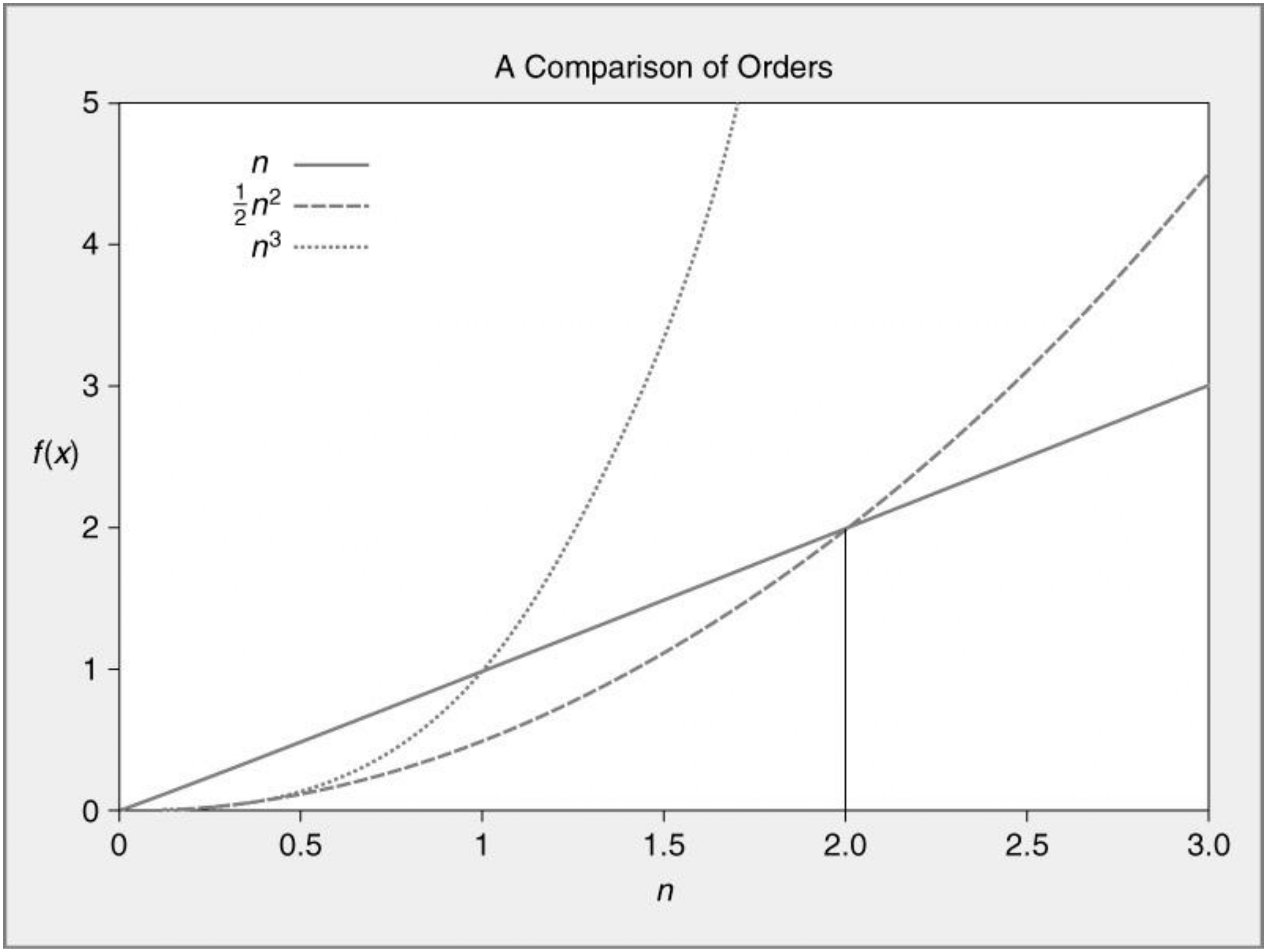
Big-O Examples

- $2n^2 + 10n + 1000 \Rightarrow O(n^2)$
  - Select $M = 4$, $N_0 = 100$
  - For $n \geq 100$
    - $4n^2 \geq 2n^2 + 10n + 1000$ is always true
  - Example $\Rightarrow$ for $n = 100$
    - $40000 \geq 20000 + 1000 + 1000$

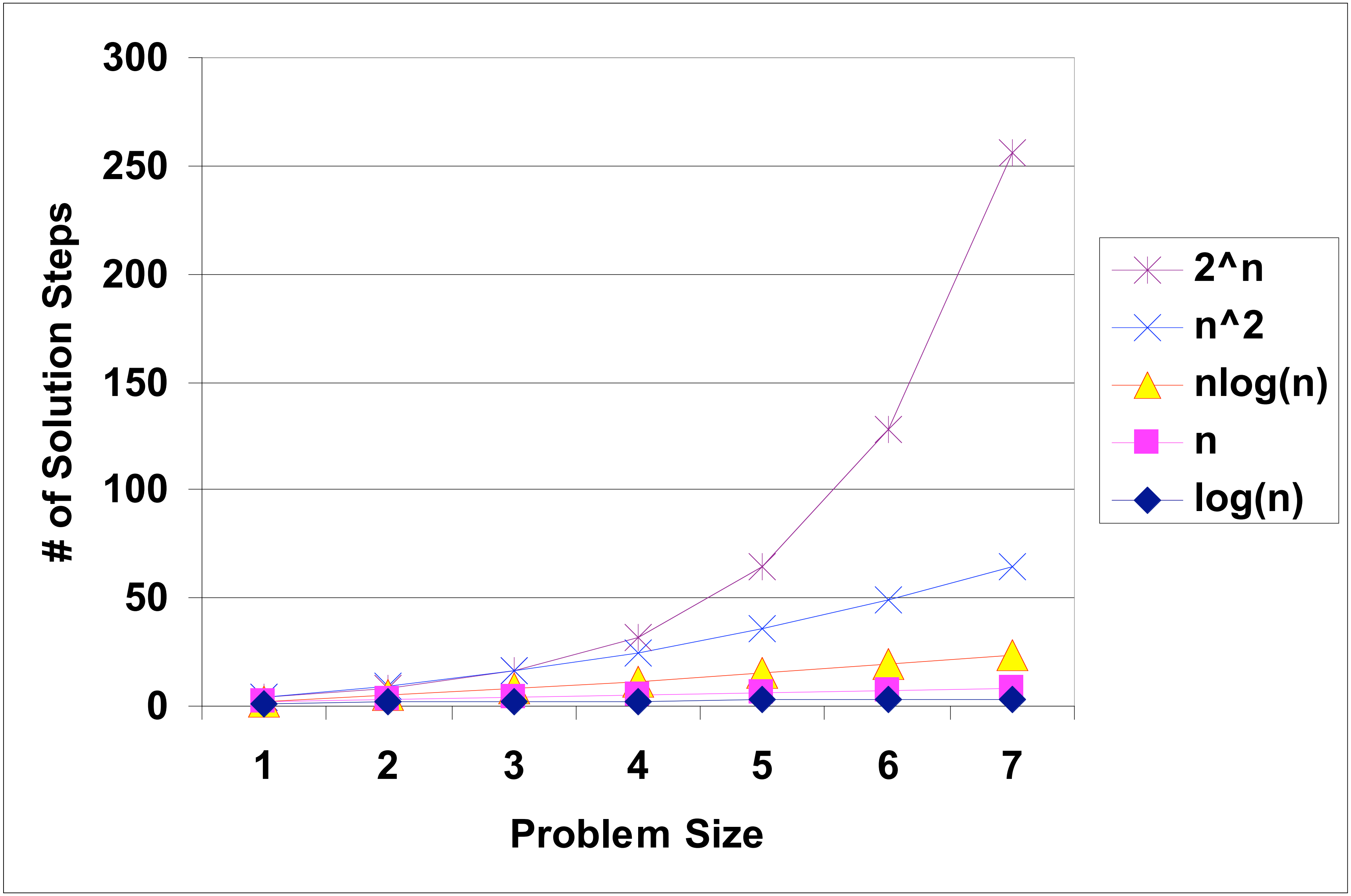
Observations

- **Big O categories**
  - **O(log(n))**
  - **O(n)**
  - **$O(n^2)$**
- **For large values of n**
  - **Any O(log(n)) algorithm is faster than O(n)**
  - **Any O(n) algorithm is faster than $O(n^2)$**
- **Asymptotic complexity is fundamental measure of efficiency**

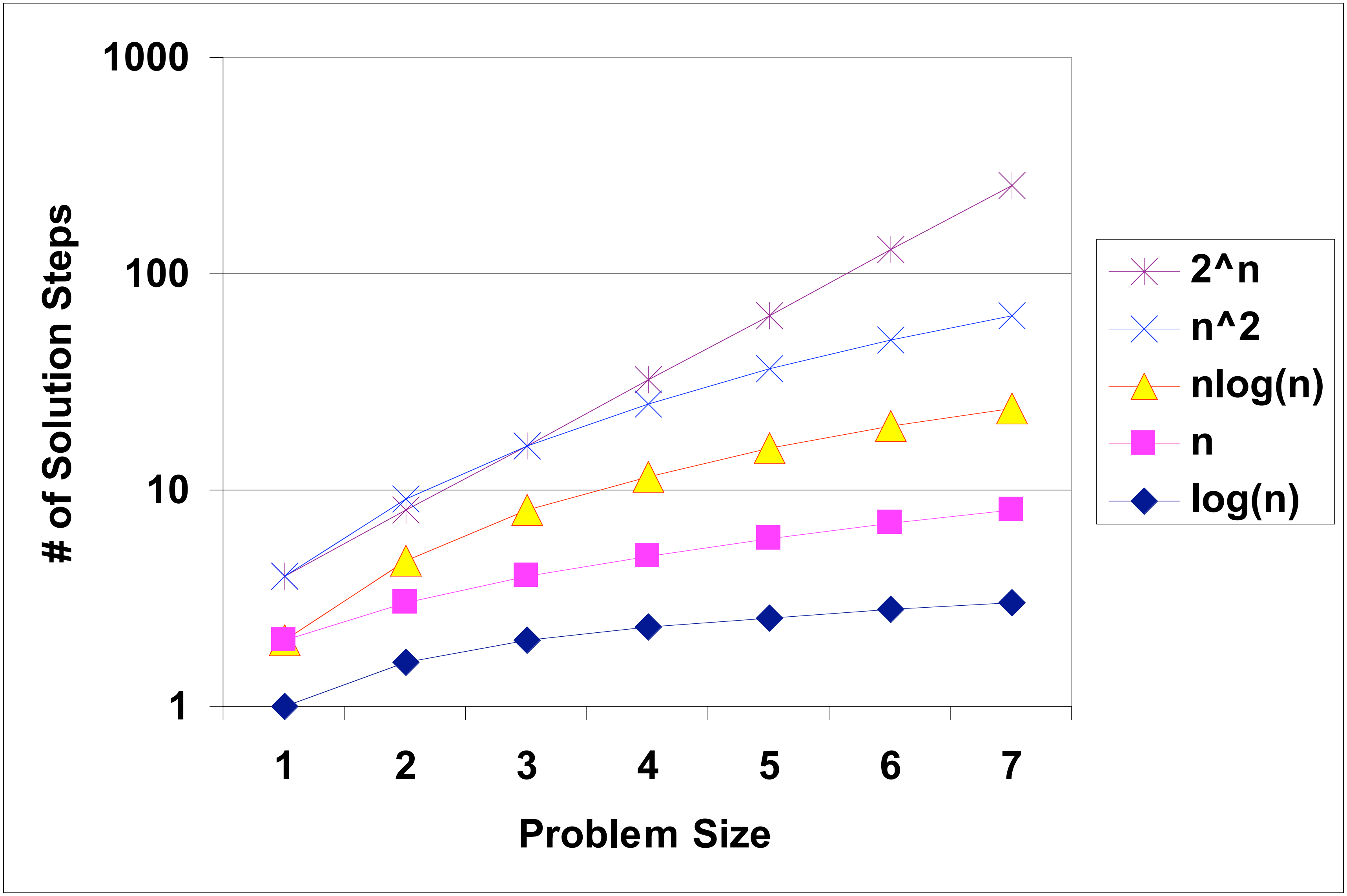
Comparison of Complexity



Complexity Category Example



Complexity Category Example

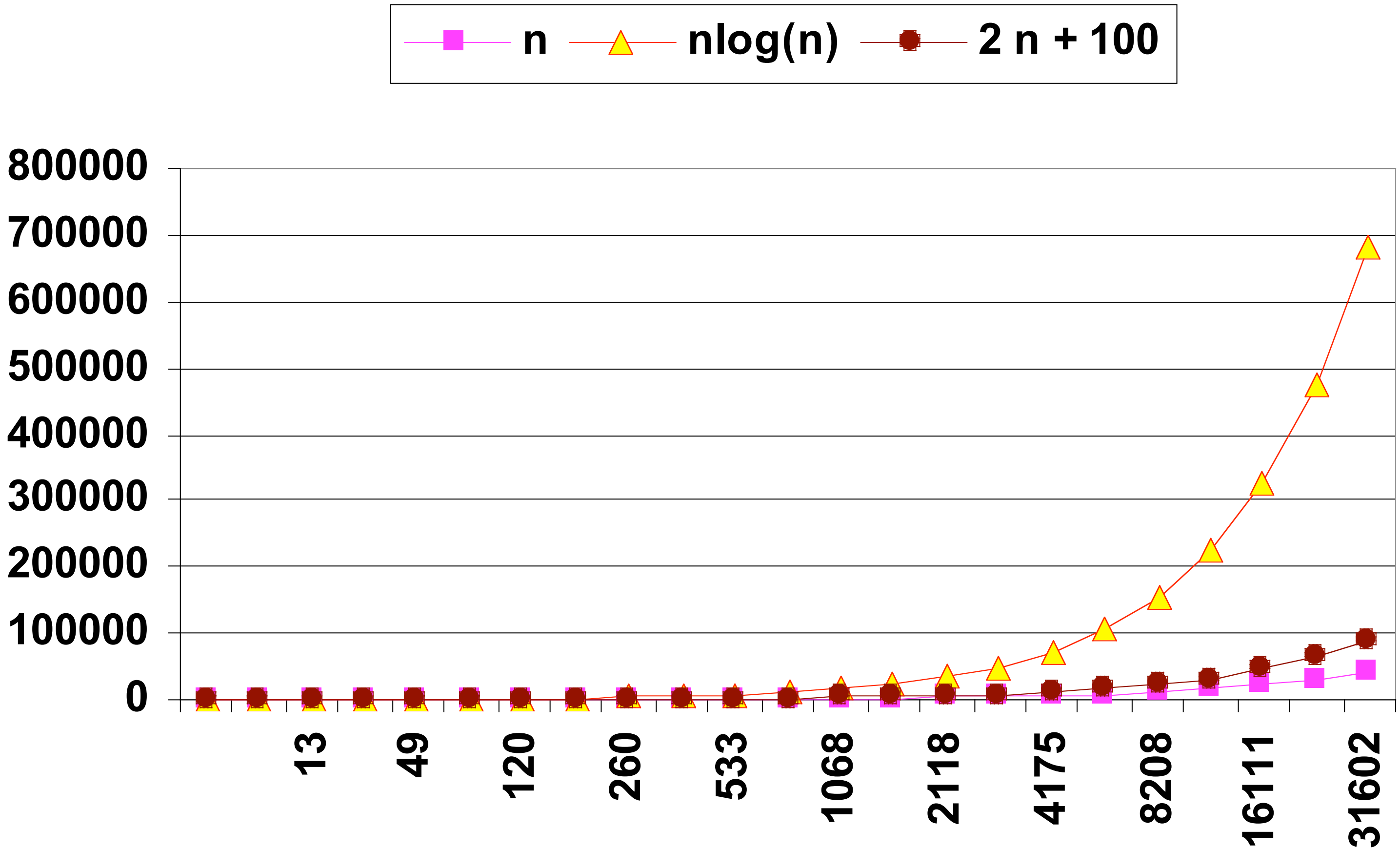


Calculating Asymptotic Complexity

- As n increases
  - Highest complexity term dominates
  - Can ignore lower complexity terms
- Examples
  - $2\ n + 100 \Rightarrow O(n)$
  - $n \log(n) + 10\ n \Rightarrow O(n\log(n))$
  - $\frac{1}{2}\ n^2 + 100\ n \Rightarrow O(n^2)$
  - $n^3 + 100\ n^2 \Rightarrow O(n^3)$
  - $1/100\ 2^n + 100\ n^4 \Rightarrow O(2^n)$

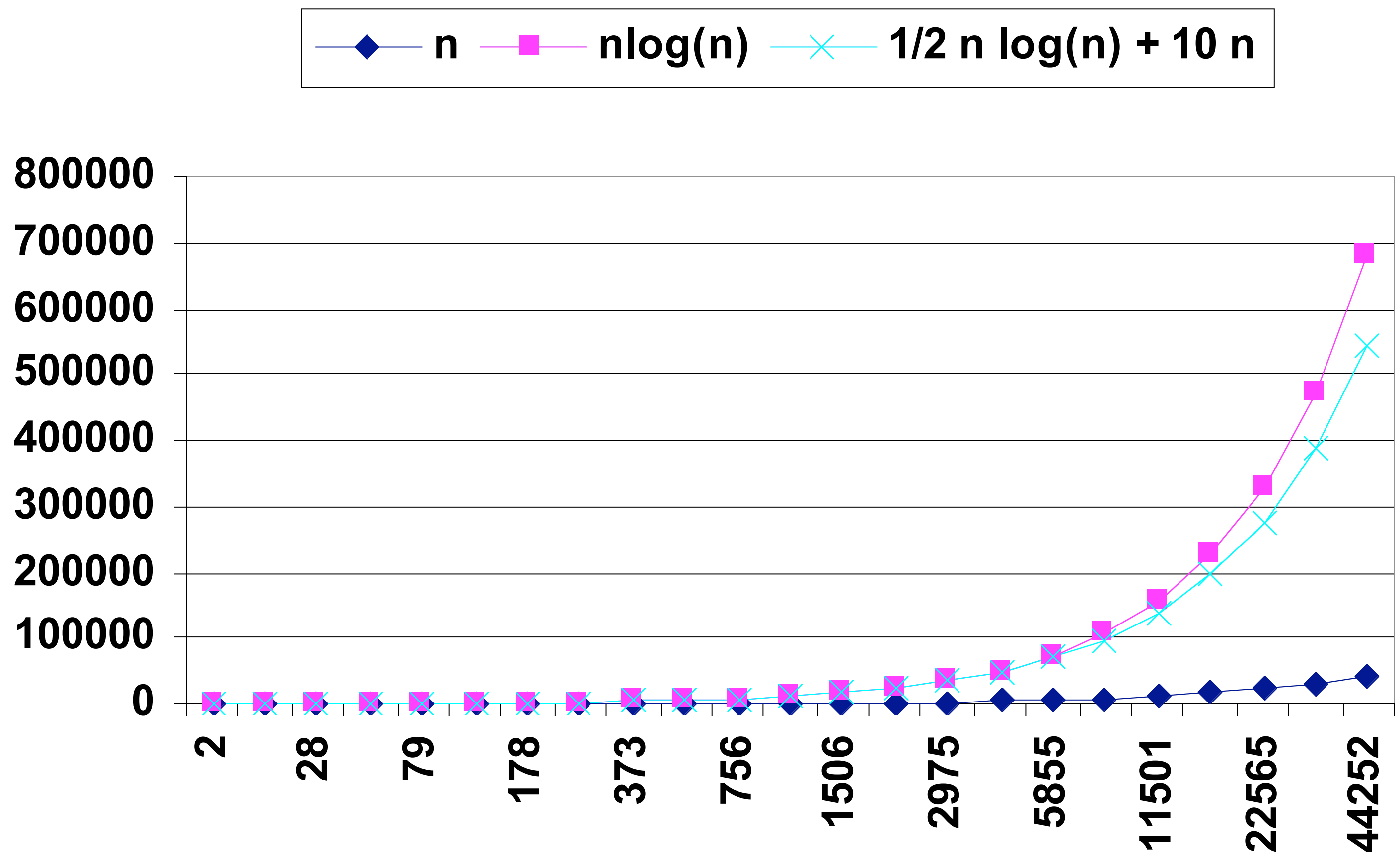
Complexity Examples

- 2n + 100 $\Rightarrow$ O(n)



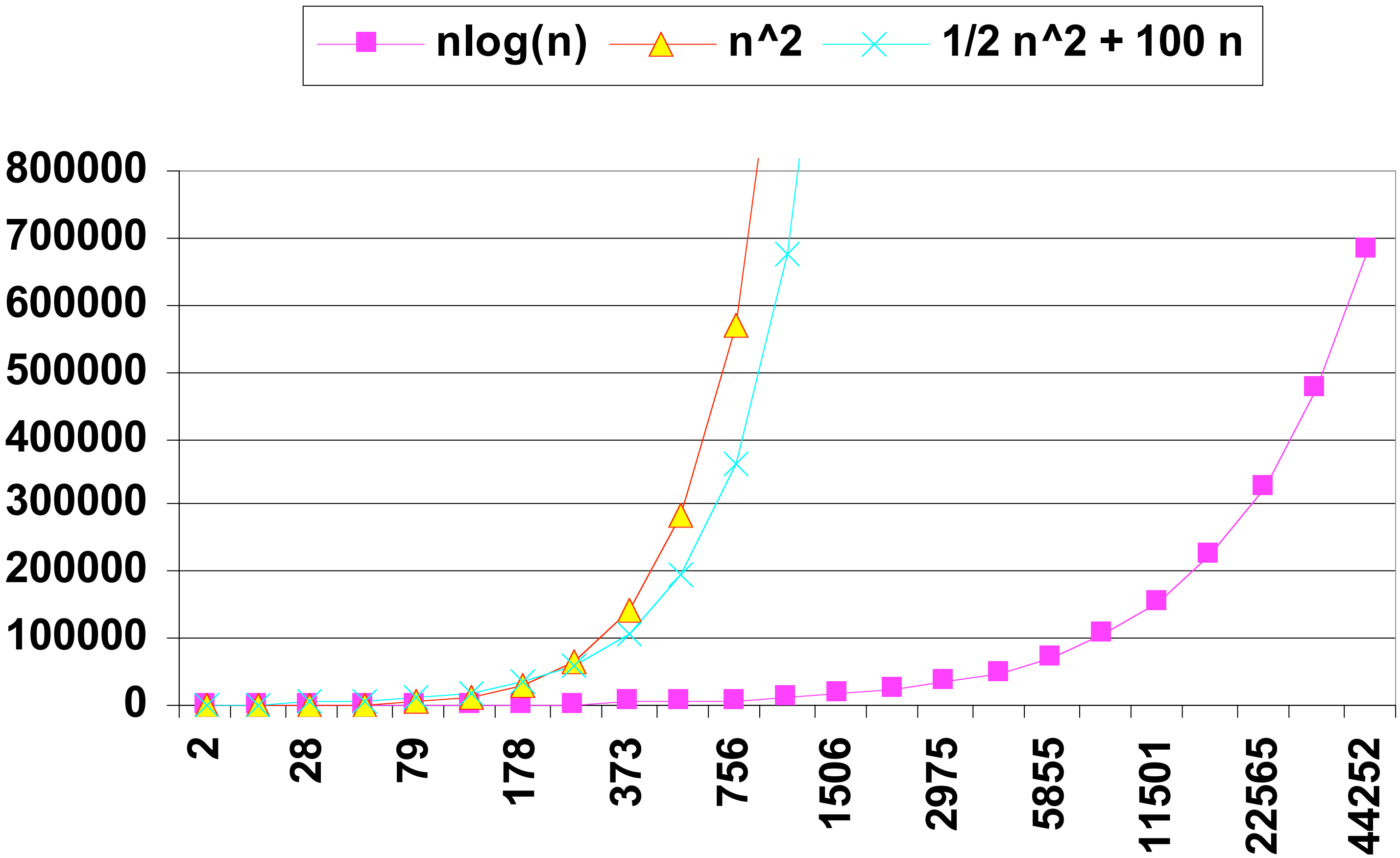
Complexity Examples

- ½ n log(n) + 10 n ⇒ O(nlog(n))



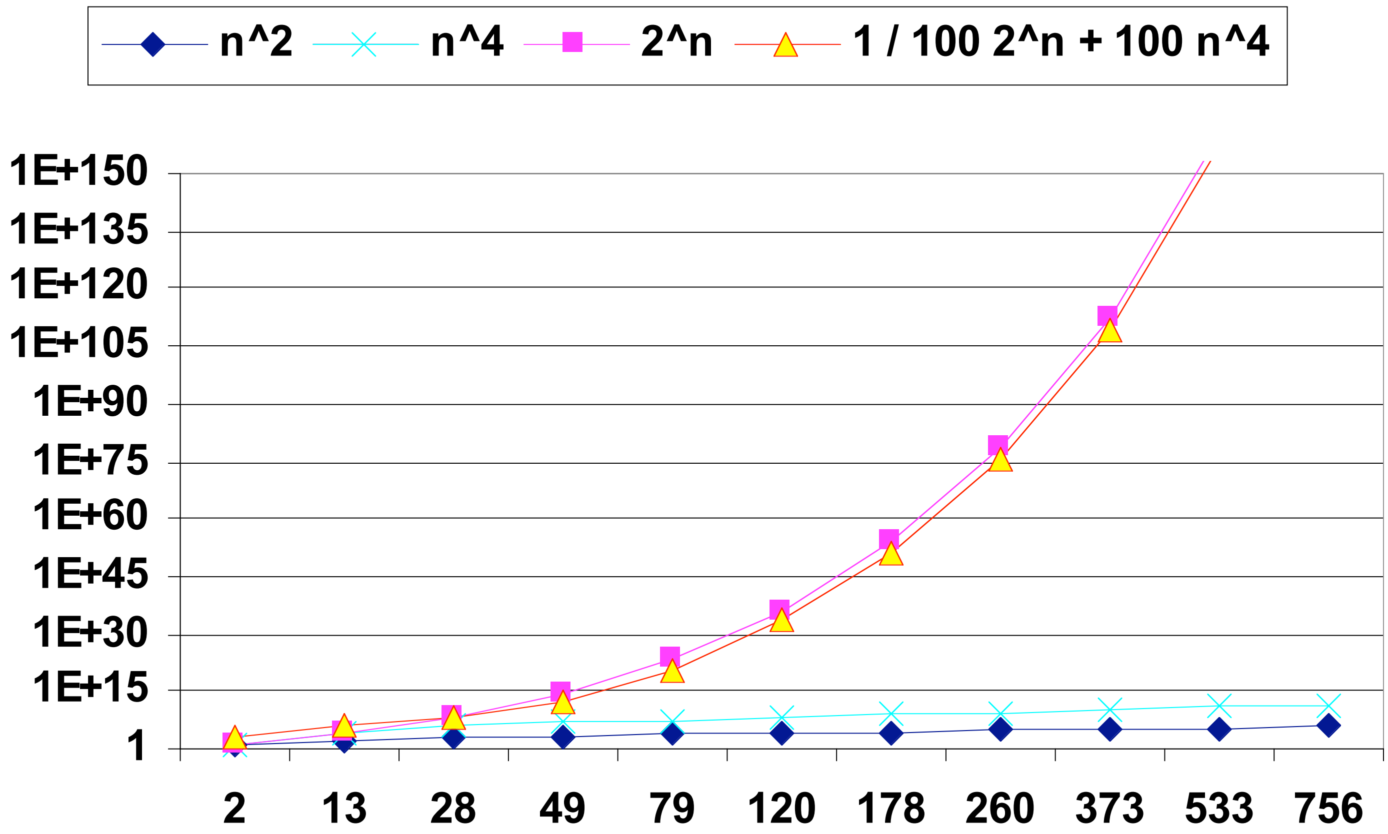
Complexity Examples

- $\frac{1}{2} n^2 + 100\, n \Rightarrow O(n^2)$



Complexity Examples

- $1/100\ 2^n + 100\ n^4 \Rightarrow O(2^n)$



Types of Case Analysis

- Can analyze different types (cases) of algorithm behavior
- Types of analysis
  - Best case
  - Worst case
  - Average case

Types of Case Analysis

- Best case
  - Smallest number of steps required
  - Not very useful
  - Example $\Rightarrow$ Find item in first place checked

Types of Case Analysis

- **Worst case**
  - **Largest number of steps required**
  - **Useful for upper bound on worst performance**
    - **Real-time applications (e.g., multimedia)**
    - **Quality of service guarantee**
  - **Example $\Rightarrow$ Find item in last place checked**

Quicksort Example

- Quicksort
  - One of the fastest comparison sorts
  - Frequently used in practice
- Quicksort algorithm
  - Pick pivot value from list
  - Partition list into values smaller & bigger than pivot
  - Recursively sort both lists

Quicksort Example

- Quicksort properties
  - Average case = O(nlog(n))
  - Worst case = $O(n^2)$
    - Pivot ≈ smallest / largest value in list
    - Picking from front of nearly sorted list
- Can avoid worst-case behavior
  - Select random pivot value

Types of Case Analysis

- Average case
  - Number of steps required for “typical” case
  - Most useful metric in practice
  - Different approaches
    - Average case
    - Expected case
    - Amortized

Approaches to Average Case

- Average case
  - Average over all possible inputs
  - Assumes some probability distribution, usually uniform
- Expected case
  - Algorithm uses randomness
  - Worse case over all possible input
  - average over all possible random values
- Amortized
  - for all long sequences of operations
  - worst case total time divided by # of operations

Amortization Example

- Adding numbers to end of array of size k
  - If array is full, allocate new array
    - Allocation cost is O(size of new array)
  - Copy over contents of existing array
- Two approaches
  - Non-amortized
    - If array is full, allocate new array of size k+1
  - Amortized
    - If array is full, allocate new array of size 2k
  - Compare their allocation cost

Amortization Example

- Non-amortized approach
  - Allocation cost as table grows from 1..n

Size (k)	1	2	3	4	5	6	7	8
Cost	1	2	3	4	5	6	7	8

  - Total cost ⇒ $n(n+1)/2$
- Case analysis
  - Best case ⇒ allocation cost = k
  - Worse case ⇒ allocation cost = k
  - Amortized case ⇒ allocation cost = $(n+1)/2$

Amortization Example

- Amortized approach
  - Allocation cost as table grows from 1..n

Size (k)	1	2	3	4	5	6	7	8
Cost	2	0	4	0	8	0	0	0

  - Total cost ⇒ $2\ (n - 1)$
- Case analysis
  - Best case ⇒ allocation cost = 0
  - Worse case ⇒ allocation cost = $2(k - 1)$
  - Amortized case ⇒ allocation cost = 2
- An individual step might take longer, but faster for any sequence of operations