1) Let $G$ be an abelian group. Let $H$ be the subset of $G$ consisting of the identity $e$ along with all elements of $G$ of order 2. Show that $H$ is a subgroup of $G$.

**proof:**

We want to show $H = \{ h \in G : |<h>| = 2 \text{ or } h = e \}$ is a subgroup of $G$. We do so by using our subgroup theorem. That is, we show $H$ is closed under the induced operation of $G$, that $H$ contains the identity $e$ of $G$, and that $\forall h \in H$, $h^{-1} \in H$.

**(closure)** Let $h, k$ be elements of $H$. If $h = e$, then $hk = ek = k$, which is in $H$. If $k = e$, then $hk = he = h$, which is in $H$. That is, if either of $h$ or $k$ are the identity, then $hk \in H$. Assume neither $h$ nor $k$ are $e$. Consider $<hk>$ which contains $e$ and $hk$. We see:

\[
(hk)^2 = (hk)(hk) \quad \text{definition of exponents}
\]

\[
= h(kh)k \quad \text{by associativity}
\]

\[
= h(hk)k \quad \text{since } G \text{ is abelian, it's operation is commutative}
\]

\[
= (hh)(kk) \quad \text{by associativity}
\]

\[
= ee \quad \text{since each of } h, k \text{ are elements in } H, \text{ and neither of } h \text{ or } k \text{ are the identity,}
\]

\[
|<h>| = 2 \text{ and } |<k>| = 2, \text{ which implies } h^2 = 2 = k^2
\]

That is, $(hk)^2 = e$. Thus $<hk> = \{ e, hk \}$ which implies $|<hk>| = 2$. That is, the order of $hk$ is 2. Thus $hk \in H$. We have shown $\forall h, k \in H$ that $hk \in H$. Therefore $H$ is closed under the induced operation from $G$.

**(identity)** The identity $e$ of $G$ is defined to be in $H$ by the definition of $H$.

**(inverses)** Let $h$ be an element in $H$. Since $h \in H$, then either $h = e$, or $|<h>| = 2$. If $h = e$ then $h$ is its own inverse as $ee = e$. Further if $h \neq e$ then $|<h>| = 2$, which implies $hh = e$. That is, the order or the cyclic subgroup generated by $h$ equals 2 implies $hh = e$. Thus $h = h^{-1}$ so $h^{-1} \in H$.

We have shown $H$ is closed under the induced operation of $G$, that $H$ contains the identity $e$ of $G$, and that $\forall h \in H$, $h^{-1} \in H$. Therefore, by our subgroup theorem, $H$ is a subgroup of $G$. 
2) Prove it is not the case that for all groups $G$ the subset $H$ of $G$ consisting of the identity $e$ along with all elements of $G$ of order 2 is a subgroup of $G$. (Be complete).

\textbf{proof:}

We show it is not the case that for all groups $G$ the subset $H$ of $G$ consisting of the identity $e$ along with all elements of $G$ of order 2 is a subgroup of $G$. We do so by counterexample. That is, we give an example of a group $G$ such that $H = \{ h \in G : \langle h \rangle = 2 \text{ or } h = e \}$ is not a subgroup of $G$.

Consider $S_3 = \{ (1), (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2) \}$, the symmetric group on three letters. Since each transposition has order 2, and each three-cycle has order 3, the set $H = \{ (1), (1\ 2), (1\ 3), (2\ 3) \}$. But $(1\ 2)(2\ 3) = (1\ 2\ 3)$ is not in $H$. Thus $H$ is not closed under the induced operation of permutation multiplication. Thus $H$ is not a subgroup of $S_3$.

We have shown it is not the case that for all groups $G$ the subset $H$ of $G$ consisting of the identity $e$ along with all elements of $G$ of order 2 is a subgroup of $G$ by use of a counterexample.
3) Let $H$ and $K$ be groups and let $G = H \times K$. Both $H$ and $K$ appear as subgroups of $G$ in a natural way.

a) What is meant by both $H$ and $K$ appear as subgroups of $G$ in a natural way?

b) Show every element of $G$ is of the form $hk$ for some $h \in H$ and $k \in K$.

c) Show $hk = kh \ \forall \ h \in H$ and $k \in K$.

d) Show $H \cap K = e$.

solution:

a) We want to show both $H$ and $K$ appear as subgroups of $G$ in a natural way. We know $H \times K = \{(h, k) \mid h \in H \text{ and } k \in K\}$. Let $e_H$ and $e_K$ be the identities for $H$ and $K$, respectively. Now consider $H \times \{e_K\} = \{(h, e_K) \mid h \in H\}$, which is a subgroup of $G$. We see $H \times \{e_K\} \subseteq H \times K$. Moreover, $H \times \{e_K\}$ can be thought of as a copy of $H$ by identifying each element $h \in H$ with the corresponding element $(h, e_K)$ in $H \times \{e_K\}$. This makes $H$ appear in $G$ in a natural way. Similarly, we consider the subgroup $\{e_H\} \times K = \{(e_H, k) \mid k \in K\}$ of $G$. Identify each $k \in K$ with the corresponding element $(e_H, k)$ in $H \times K$. Then $K$ appears as a subgroup of $G$ in a natural way.

b) We want to show every element of $G$ is of the form $hk$ for some $h \in H$ and $k \in K$. Let $(h, k)$ be an element of $G$, for some $h \in H$ and $k \in K$. Then $(h, k) = (h, e_K)(e_H, k)$, by the definition of the operation in $H \times K$ and since $h e_H = h$, and $e_K k = k$, by definition of the identity of a group. Identifying $h$ with $(h, e_K)$ and identifying $k$ with $(e_K, k)$, as described in a), we see that $(h, k) = (h, e_K)(e_H, k)$ is identified with $hk$. This shows that every element of $G$ is of the form $hk$ for some $h \in H$ and $k \in K$.

c) Let $h \in H$ and $k \in K$. Consider $(h, k)$ in $H \times K$. We see $(h, k) = (e_H, k)(h, e_K)$, by the definition of the operation in $H \times K$. Using the identification we have described, $(h, k)$ corresponds to $kh$. But in b) we showed $(h, k)$ corresponded to $hk$. Thus $kh = hk$ under this identification.

d) $H$ appears in $G$ as $H \times \{e_K\} = \{(h, e_K) \mid h \in H\}$ and $K$ appears in $G$ as $\{e_H\} \times K = \{(e_H, k) \mid k \in K\}$. We see $H \times \{e_K\} \cap \{e_H\} \times K = (e_H, e_K)$ which is identified with $(e_H, e_K)$, the identity in $H \times K$. This shows $H \cap K = e$ where $e$ is the identity of $G$ under the identification.