1. Determine whether \( f : \mathbb{R} \rightarrow \mathbb{R} \) defined by \( f(x) = -x^3 \) is a permutation of \( \mathbb{R} \). Give a detailed proof of your answer.

We claim \( f \) is a permutation of \( \mathbb{R} \).

**proof of claim:**

To show \( f : \mathbb{R} \rightarrow \mathbb{R} \) defined by \( f(x) = -x^3 \) is a permutation of \( \mathbb{R} \) we show \( f \) is one-to-one and \( f \) maps onto \( \mathbb{R} \).

(one-to-one) Let \( x_1, x_2 \in \mathbb{R} \) such that \( f(x_1) = f(x_2) \). Then,

\[
\begin{align*}
f(x_1) &= f(x_2) \\
-(x_1)^3 &= -(x_2)^3 & \text{defn. of } f \\
(x_1)^3 &= (x_2)^3 & \text{mul. by -1} \\
x_1 &= x_2 & \text{take cube root of each side*} \\
\end{align*}
\]

That is, \( f(x_1) = f(x_2) \) implies \( x_1 = x_2 \). We have shown \( \forall x_1, x_2 \in \mathbb{R} \) that \( f(x_1) = f(x_2) \) implies \( x_1 = x_2 \). Thus \( f \) is a one-to-one, by definition of one-to-one.

(onto \( \mathbb{R} \)) Let \( y \in \mathbb{R} \). Consider, \( x = -\sqrt[3]{y} \in \mathbb{R} \), which exists by the definition of the cube root function. We see:

\[
\begin{align*}
f(x) &= f(-\sqrt[3]{y}) & \text{by substitution} \\
&= -(\sqrt[3]{y})^3 & \text{by defn. of } f \\
&= -y & \text{since the cube root function and the cube function are inverses of one another**} \\
&= y \\
\end{align*}
\]

That is, \( f(x) = y \). We have shown \( \forall y \in \mathbb{R} \) that \( \exists x \in \mathbb{R} \), namely \( x = -\sqrt[3]{y} \) such that \( f(x) = y \). Thus \( f \) maps onto \( \mathbb{R} \), by definition of onto \( \mathbb{R} \).

We have shown \( f \) is one-to-one and \( f \) maps onto \( \mathbb{R} \). Therefore \( f \) is a permutation of \( \mathbb{R} \).

* Here we are using the cube root function is one-to-one

** Here we are using the cube root function and the cube function are inverses

Remark. Recognizing * and ** leaves “begging the question”, how can we show * and **? Think outside the course.
2. Let $A$ be a set. Let $B$ be a subset of $A$ and let $b$ be one particular element of $B$. Determine whether 

$$J = \{ \sigma \in S_A \mid \sigma(b) = b \}$$

is sure to be a subgroup of $S_A$ under the induced operation. Give a detailed proof of your answer.

We claim $J$ will always be a subgroup of $S_A$.

proof of claim:

We prove our claim by using our subgroup theorem. That is, we show that $J$ is closed under permutation multiplication, the induced operation from $S_A$, the identity $e$ of $S_A$ is in $J$, and for all $\sigma \in J$, $\sigma^{-1} \in J$.

(closure) Let $\alpha, \beta \in J$. Since $\alpha \in J$, $\alpha(b) = b$. Similarly since $\beta \in J$, $\beta(b) = b$, where $b$ is the one particular element of $B$. Consider $\alpha \beta$.

$$(\alpha \beta)(b) = \alpha(\beta(b)) \quad \text{by definition of permutation multiplication}$$

$$= \alpha(b) \quad \text{since } \alpha \in J$$

$$= b \quad \text{since } \beta \in J$$

That is, $(\alpha \beta)(b)$, which implies $\alpha \beta \in J$, by definition of $J$. We have shown $\forall \alpha, \beta \in J$ that $\alpha \beta = J$. Thus $J$ is closed under permutation multiplication.

(identity) Let $e$ be the identity of $S_A$. So, $e(a) = a \forall a \in A$, by definition of the identity of $S_A$. In particular $e(b) = b$ the one particular element of $B$. Thus, $e \in J$, by definition of $J$.

(inverses) Let $\sigma \in J$. $\sigma$ is in $S_A$ so $\sigma^{-1}$ exists in $S_A$. Moreover, $\sigma \in J$ implies $\sigma(b) = b$. This implies

$$\sigma^{-1}(\sigma(b)) = \sigma^{-1}(b), \quad \text{applying } \sigma^{-1} \text{ to each side.}$$

implies $$(\sigma^{-1} \sigma)(b) = \sigma^{-1}(b), \quad \text{definition of permutation multiplication}$$

implies $e(b) = \sigma^{-1}(b)$ since $\sigma^{-1}$ is the inverse of $\sigma$

implies $b = \sigma^{-1}(b)$ definition of $e$

That is: $\sigma^{-1}(b) = b$. So $\sigma^{-1} \in J$. We have shown $\forall \sigma \in J$, $\sigma^{-1} \in J$ so $J$ contains the inverse of each of its elements.

Therefore, by our subgroup theorem, $J$ is a subgroup of $S_A$. 

3. Consider the following problem and then answer the questions that are asked. (Note: You are not being asked to give a proof.)

Let $G$ a group. Prove that the permutations $\sigma_a : G \to G$ where $\sigma_a(x) = xa$ for $a \in G$ and $x \in G$ do form a group isomorphic to $G$.

a) What conditions must $\sigma_a$ satisfy in order to be a permutation.

$\sigma_a : G \to G$ must be a one-to-one function mapping onto $G$.

b) Write set builder notation for the set of permutations the problem is asking you to prove is isomorphic to $G$. Give the set the name $J$.

We see that $\sigma_a : G \to G$ is an element of $S_G$.

Here: $J = \{ \alpha \in S_G \mid \exists a \in G \text{ with } \alpha = \sigma_a \}$.

$c) What must be done in order to show $J$ is a group?

Since $J \subseteq S_G$ we can use our subgroup theorem. That is, we show $J$ is closed under the induced operation (permutation multiplication), that the identity $e$ of $S_G$ is in $J$, and for each $\alpha \in J$, $\alpha^{-1} \in J$.

d) Give a map (with an appropriate name) from $G$ to $J$ that will show $G$ and $J$ are isomorphic.

$\phi : G \to J$ defined by $\phi(a) = \sigma_a$ can be shown to be an isomorphism.