1. **State:** The definition of a permutation of a set $A$. Be complete.

   \[
   \text{Defn: A permutation of a set } A \text{ is a function } \varphi : A \rightarrow A \text{ that is both one-to-one and onto.}
   \]

2. **State:** Cayley's Theorem.

   \[
   \text{Cayley's Theorem: Every group is isomorphic to a group of permutations.}
   \]

3. Consider $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 8 & 1 & 2 & 4 & 10 & 7 & 5 & 9 & 6 \end{pmatrix}$.

   a) Find all the orbits of $\alpha$. Write your answer in correct notation (there are two correct notations).
   
   \[
   \begin{align*}
   [1] &= \{1, 3\} & [7] &= \{7\} \\
   [6] &= \{6, 10\}
   \end{align*}
   \]

   (Remark: Alternate notation "over bar" instead of $[ ]$.

   e.g. $T = \{1, 3\}$ etc.)

   b) Express $\alpha$ as a product of disjoint cycles.

   \[
   \alpha = (1 \ 3)(2 \ 8 \ 5 \ 4)(6 \ 10)
   \]

4. Compute this product of cycles that are in $S_{10}$.

   \[
   (2 \ 6 \ 4 \ 9)(3 \ 9 \ 7 \ 10 \ 5)(6 \ 9) = (2 \ 6 \ 7 \ 10 \ 5 \ 3)(4 \ 9)
   \]
5. Let \( A \) be a set. Let \( B \) be a subset of \( A \) and let \( b \) be one particular element of \( B \). Determine whether the given subset \( J \) is sure to be a subgroup of \( S_A \) under the induced operation.

a) Give a detailed proof of your answer. Start by clearly stating whether you are proving \( J \) is or is not a subgroup. \( J = \{ \sigma \in S_A \mid \sigma[B] = [B] \} \)

We claim \( J \) is a subgroup of \( S_A \).

Proof of claim:

We use our subgroup theorem to show \( J \subseteq S_A \).

(closed under perm. mult). Let \( \alpha, \beta \in J \). Consider \( \alpha \beta \).

We show \( \alpha \beta \in J \). Let \( x \in B \). Then \( (\alpha \beta)(x) = \alpha(\beta(x)) \), defn. perm. mult., \( = \alpha(y) \), (where \( y \in B \) since \( \beta \in J \)), \( = z \), where \( z \in B \) since \( \alpha \in J \). That is, \( (\alpha \beta)(x) \in B \) \( \forall x \in B \). Thus \( (\alpha \beta)[B] \subseteq B \).

Let \( z \in B \). Since \( \alpha \in J \), \( \alpha[B] = B \), so \( y \in B \) \( \exists \) \( y \in B \) such that \( \beta(x) = y \). Then \( (\alpha \beta)(x) = \alpha(\beta(x)) = \alpha(y) = z \). This shows \( B \subseteq (\alpha \beta)[B] \). \( \alpha \beta \in J \).

(identity is in \( J \)). Let \( e \) be the identity of \( S_A \).

Let \( x \in B \), then \( e(x) \in B \). Thus \( \forall x \in B \), \( e(x) \in B \) so \( e[B] = B \).

Moreover \( \forall x \in B \), \( e(x) = x \), so \( B \subseteq e[B] \).

Thus \( e[B] = B \) and \( e \in J \).

(inverses are in \( J \)) Let \( \alpha \in J \). Consider \( \alpha^{-1} \).

Since \( \alpha \in J \), \( \alpha[B] = B \).

Thus \( \alpha^{-1}(y) = \alpha^{-1}(\alpha(x)) = x \).

Further ...

(b) Determine whether \( J = \{ \sigma \in S_A \mid \sigma(b) \in B \} \) is sure to be a subgroup of \( S_A \) under the induced operation. Briefly explain your answer using completer well written sentences.

We claim \( J \) is not necessarily a subgroup of \( S_A \).

Proof of claim:

Consider the counter example: \( A = \{1, 2, 3\} \), \( B = \{1, 2, 3\} \) and \( b = 1 \). We see \( \alpha = (1 \ 2 \ 3) \in J \).

Also \( \beta = (1 \ 2 \ 3) \in J \) but \( \alpha \beta = (1 \ 2 \ 3)(1 \ 2 \ 3) = (1 \ 2 \ 3) \neq J \) since \( (\alpha \beta)(1) = 3 \notin B \).