1. Define an isomorphism of two algebraic structures as presented. (I recommend using the same notation as we did in class). The definition must be complete.

**Defn:** Let \( S, \cdot \) and \( T, \circ \) be binary algebraic structures. An isomorphism of \( S \) with \( T \) is a one-to-one function \( \psi \) mapping \( S \) onto \( T \) such that for all \( x_1, x_2 \in S \),
\[
\psi(x_1 \cdot x_2) = \psi(x_1) \circ \psi(x_2)
\]

2. The map \( \phi : \mathbb{Q} \to \mathbb{Q} \) defined by \( \phi(x) = 5x - 1 \) for \( x \in \mathbb{Q} \) is one-to-one.

   a) Prove \( \phi \) maps onto \( \mathbb{Q} \).

We want to show \( \forall y \in \mathbb{Q} \exists x \in \mathbb{Q} \) such that
\[
\phi(x) = y.
\]
Let \( y \in \mathbb{Q} \). Consider \( x = (y + 1)/5 \)
which is in \( \mathbb{Q} \) since \( \mathbb{Q} \) is closed under + and 1/.
We see \( \phi(x) = \phi( (y+1)/5 ) = 5( (y+1)/5 ) - 1 = y \).
That is, \( \phi(x) = y \). We have shown \( \forall y \in \mathbb{Q} \exists x \in \mathbb{Q} \)
such that \( \phi(x) = y \). Therefore \( \phi \) maps onto \( \mathbb{Q} \).

b) (Assume \( \phi \) is a one-to one function mapping onto \( \mathbb{Q} \). ) Give the definition of a binary operation \( \cdot \) on \( \mathbb{Q} \) such that \( \phi \) is an isomorphism mapping \( \langle \mathbb{Q}, \cdot \rangle \) onto \( \langle \mathbb{Q}, \ast \rangle \). Indicate your reasoning.

We want to find \( \cdot \) so that \( \forall x_1, x_2 \in \mathbb{Q} \)
\[\text{equation 1.} \quad \phi(x_1 \cdot x_2) = \phi(x_1) \ast \phi(x_2) \quad \text{where} \quad \phi(x) = 5x - 1, \]
from a) we see for \( y \in \mathbb{Q} \) letting \( x = (y+1)/5 \)
we have \( \phi(x) = y \). Let \( x_1, x_2 \in \mathbb{Q} \) and let
\[ a = 5x_1 - 1 \quad \text{and} \quad b = 5x_2 - 1. \]
Note \( a, b \in \mathbb{Q} \),
further \( x_1 = (a+1)/5 \) and \( x_2 = (b+1)/5 \),
and \( \phi(x_1) = a, \phi(x_2) = b \). So the RHS of eq 1 becomes \( a \ast b \).
The LHS of eq 1 is
\[
\phi(x_1 \cdot x_2) = \phi( \frac{a+1}{5} - \frac{b+1}{5} ) = \phi( \frac{(a+1)(b+1)}{25} ) = \frac{5}{25} \frac{(a+1)(b+1)}{25} - 1 = \frac{1}{5} (a+1)(b+1) - 1.
\]
Define \( \ast \) by:
\[
a \ast b = \frac{1}{5} (a+1)(b+1) - 1.
\]
3. Let $S$ be the set of all 2 by 2 matrices with determinant 1, whose entries come from $\mathbb{R}$.

a) Give two examples of elements in $S$.

Many examples: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$

b) Consider $S$ under matrix addition $+_m$. Is $<S, +_m>$ a group? No

Give a brief but compelling explanation.

Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ then $A, B \in S$ but, $A +_m B = \begin{pmatrix} 3 & 1 \\ 5 & 4 \end{pmatrix}$

We see $\det(A +_m B) = 3(4) - 5 = 7$ so $A +_m B \notin S$.

$<S, +_m>$ is not a group because $+_m$ is not a binary operation. That is $S$ is not closed under $+_m$.

c) Consider $S$ under matrix multiplication $\cdot_m$. Is $<S, \cdot_m>$ a group? Yes

Give a brief but compelling explanation.

Let $A, B \in S$. Then $\det(A \cdot_m B) = (\det A)(\det B) = 1$. So $AB \in S$, making $\cdot_m$ a binary operation on $S$.

We know matrix mul. is associate so restricted to $S$ it is associative.

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has $\det(I) = 1$ so $I$, the multiplicative identity for 2x2 matrices, is in $S$.

We know every matrix $A$ with nonzero det. has an inverse. Let $A \in S$. Then $\det A = 1$. Further $1 = \det(I) = \det(AB^{-1}) = (\det A)(\det(A^{-1})) = \det(A^{-1})$.

That is, $\det(A^{-1}) = 1$. So $A^{-1} \in S$.

The group properties are satisfied so $<S, \cdot_m>$ is a group.