1. Show that $[1, 3]$ and $[5, 25]$ have the same cardinality by giving a formula for a one-to-one function $f$ mapping $[1, 3]$ onto $[5, 25]$. You must prove the given function is one-to-one and onto $[5, 25]$.

Proof:

We want to show $[1, 3]$ and $[5, 25]$ have the same cardinality by giving a formula for a one-to-one function $f$ mapping $[1, 3]$ onto $[5, 25]$.

Consider the function $f : [1, 3] \rightarrow \mathbb{R}$ defined by $f(x) = 10x - 5$. $f$ has the correct domain. In order to show it is a one-to-one function mapping $[1, 3]$ onto $[5, 25]$, we show $f$ is one-to-one and maps onto $[5, 25]$.

(1-1) Let $x_1, x_2 \in [1, 3]$ such that $f(x_1) = f(x_2)$. Then $10x_1 - 5 = 10x_2 - 5$, which implies $10x_1 = 10x_2$ so $x_1 = x_2$. We have shown that $f(x_1) = f(x_2)$ implies $x_1 = x_2$. This shows $f$ is a one-to-one function.

(onto). Let $y \in [5, 25]$. Consider $x = \frac{(y+5)}{10}$. We see that:

$$f(x) = f\left(\frac{(y+5)}{10}\right) = 10\left(\frac{(y+5)}{10}\right) - 5 = y + 5 - 5 = y.$$ That is, $\forall y \in [5, 25]$ we have have found an $x$ in $\mathbb{R}$ such that $f(x) = y$.

However, it remains to demonstrate that $x \in [1, 3]$. We took $y \in [5, 25]$ which implies $5 \leq y \leq 25$. Now $f(x) = y$ implies $5 \leq f(x) \leq 25$. That is, $5 \leq 10x - 5 \leq 25$ which implies $10 \leq 10x \leq 30$ so $1 \leq x \leq 3$. That is $x \in [1, 3]$. $\forall y \in [5, 25]$ we have found an $x \in [1, 3]$, namely $x = \frac{(y+5)}{10}$, such that $f(x) = y$, so $f$ maps onto $[5, 25]$.

We have given a function $f : [1, 3] \rightarrow [5, 25]$ that is one-to-one and onto $[5, 25]$. Therefore $[1, 3]$ and $[2, 25]$ have the same cardinality.
2. Determine whether the relation \( x R y \) in \( \mathbb{R} \) if \(| x | = | y |\) is an equivalence relation. Prove your answer. If \( R \) is an equivalence relation on \( \mathbb{R} \), give the partition arising from the equivalence relation.

Prove the relation \( x R y \) in \( \mathbb{R} \) if \(| x | = | y |\) is an equivalence relation.

proof:

We want to show \( R \) is an equivalence relation on \( \mathbb{R} \). We do so by showing \( R \) satisfies the reflexive, symmetric and transitive properties.

Let \( x \in \mathbb{R} \). Then \(| x | = | x |\). That is, the absolute value of a number equals the absolute value of itself. We have shown \( x \in \mathbb{R} \) that \( x R x \) which shows that \( R \) satisfies the reflexive property.

Let \( x, y \in \mathbb{R} \) such that \( x R y \). Then \(| x | = | y |\), which implies \(| y | = | x |\) since an absolute value in this case is a real number and equality of real numbers is symmetric. Thus \( y R x \). We have shown \( \forall x, y \in \mathbb{R} \) that \( x R y \) implies \( y R x \). Hence \( R \) satisfies the symmetric property.

Let \( x, y, z \in \mathbb{R} \) such that \( x R y \) and \( y R z \). Then \(| x | = | y |\) and \(| y | = | z |\), which implies \(| x | = | z |\) since an absolute value in this case is a real number and equality of real numbers is transitive. We have shown \( \forall x, y, z \in \mathbb{R} \) that \(| x | = | y |\) and \(| y | = | z |\) together imply \(| x | = | z |\). Hence \( R \) satisfies the transitive property.

We have shown \( R \) satisfies the reflexive, symmetric and transitive properties therefore \( R \) is an equivalence relation on \( \mathbb{R} \).

Note that the equivalence class containing 0 is \([0] = \{0\}\) and \( \forall x \in \mathbb{R}, x \neq 0 \) the equivalence class containing x is \([x] = \{x, -x\}\). So the partition that arises from this equivalence relation is:

\[ \{ \{ 0 \} \} \cup \{ \{ x, -x \} \mid x \in \mathbb{R}, x > 0 \} \].
3. Determine whether the relation $x R y$ in $\mathbb{R}$ if $|x - y| \leq 3$ is an equivalence relation. Prove your answer. If $R$ is an equivalence relation on $\mathbb{R}$, give the partition arising from the equivalence relation.

Prove the relation $x R y$ in $\mathbb{R}$ if $|x - y| \leq 3$ is not an equivalence relation.

proof:

To prove the relation $x R y$ in $\mathbb{R}$ if $|x - y| \leq 3$ is not an equivalence relation it suffices to show it is not the case that $R$ satisfies the reflexive, symmetric, and transitive properties. To do so, we need only show $R$ fails to satisfy one of the properties. We show $R$ is not transitive.

Consider 7, 4, 1, which are elements of $\mathbb{R}$. We see $|7 - 4| = 3 \leq 3$, so $7 R 4$. Also, $|4 - 1| = 3 \leq 3$, so $4 R 1$. But $|7 - 1| = 6 > 3$, (that is 6 is not less than or equal to 3). So 7 is not related to 1. We have found elements of $\mathbb{R}$, namely 7, 4, 1 such that $7 R 4$ and $4 R 1$ but it is not the case that $7 R 1$. This shows $R$ does not satisfy the transitive property.

Therefore $R$ is not an equivalence relation.