DAY 38: Exam II Friday on sections 8, 9, 10, 11, 13

Defs and Theorems: permutation, Cayley's Theorem, Alternating group of degree $n$, even/odd perm., Lagrange's Theorem, isomorphism, homomorphism, $\text{Ker} (\Psi)$ where $\Psi$ is a homomorphism.

Ideas about $\text{Ker} (\Psi): \quad \Psi: G \rightarrow G'$

If $\Psi$ is a homomorphism then $\Psi(e) = e'$ where $e$ is the identity of $G$ and $e'$ is the identity of $G'$.

$\text{Ker} (\Psi) = \{ x \in G \mid \Psi(x) = e' \}$ defn.

We know from the four part theorem on properties of homomorphisms ($e \mapsto e'$, $\Psi(a^{-1}) = (\Psi(a))^{-1}$, $\Psi[H] \subseteq G'$ and $\Psi^{-1}[H'] \subseteq G$) that $\Psi^{-1}[H']$ is a subgroup of $G$ ($G$ not $G'$) (the inverse image of $H'$

That is, $\Psi^{-1}[H'] = \{ x \in G \mid \Psi(x) \in H' \}$ is a subgroup of $G$ ($G$ not $G'$)

Since $\{e'\}$ is a subgroup of $G'$ it follows directly from the theorem that $\Psi^{-1}[\{e'\}] = \text{Ker} (\Psi)$ is a subgroup of $G$. 
ex: Let \( \psi \) be a homomorphism of a group \( G \) into a group \( G' \). Prove \( \ker(\psi) \) is a subgroup of \( G \).

**proof:**

We will show \( \ker(\psi) = \{ x \in G \mid \psi(x) = e' \} \)

where \( e' \) is the identity in \( G' \) is a subgroup of \( G \) by using our subgroup theorem.

1. **Closure.** Let \( x, y \in \ker(\psi) \). Consider \( xy \).
   
   We see \( \psi(xy) = \psi(x)\psi(y) \) since \( \psi \) is a homomorphism.
   
   That is, \( \psi(xy) = e' \) which implies \( xy \in \ker(\psi) \),
   
   by defn. of \( \ker(\psi) \).

2. **Identity.** Let \( e \) be the identity in \( G \). Let \( a \in G \). Then, \( \psi(a) = \psi(ae) \) defn identity
   
   That is: \( \psi(a) = \psi(a)\psi(e) \), which implies
   
   (multiply both sides by \( \psi(a)^{-1} \)) \( e' = \psi(e) \).
   
   So \( e \in \ker(\psi) \), as desired.

3. **Inverse.** Let \( x \in \ker(\psi) \). Then \( \psi(x) = e' \).
   
   Further \( xx^{-1} = e \) implies \( \psi(xx^{-1}) = \psi(e) = e' \)
   
   (from above). But \( \psi(xx^{-1}) = \psi(x)\psi(x^{-1}) \),
   
   by the homomorphism property. Thus
   
   \( \psi(x)\psi(x^{-1}) = e' \), which implies
   
   (multiply each side on the left by \( \psi(x)^{-1} \)) \( \psi(x^{-1}) = (\psi(x))^{-1} \).
   
   That is, \( \psi(x^{-1}) = (e')^{-1} \) so \( \psi(x^{-1}) = e' \).
   
   Thus \( x^{-1} \in \ker(\psi) \). We have shown \( \forall x \in \ker(\psi), \ x^{-1} \in \ker(\psi) \).
   
   By our subgroup theorem, \( \ker(\psi) \) is a subgroup of \( G \).
ex:  
   a) define $A_n$  
   b) Prove $A_n$ is a subgroup of $S_n$. (with using defn) (of S.G, or S.G. thm).
   proof:
   Let $\phi : S_n \to \mathbb{Z}_2$ be defined by
   \[
   \phi(\sigma) = \begin{cases} 
   0 & \text{if } \sigma \text{ is even} \\
   1 & \text{if } \sigma \text{ is odd}. 
   \end{cases}
   \]
   We can show $\phi$ is a homomorphism. (this entail?)
   Further, $\ker(\phi) = \{ \sigma \in S_n \mid \phi(\sigma) = 0 \} = A_n$.
   The kernel of a homomorphism is a subgroup of its domain. Therefore $A_n \leq S_n$.

ex: Let $\phi : GL(n, \mathbb{R}) \to \langle \mathbb{R}^*, \cdot \rangle$ be defined by $\phi(A) = \det(A)$.

Note: we can "easily" show $\phi$ is a homomorphism using the matrix algebra property $\det(AB) = \det A \cdot \det B$.

Q: What is $\ker(\phi)$?