DAY 16: Start studying in §6

pg 66: 1-11, 17-29, 33-37, 42, 44, 45, 49, 50

Defn: Let $G$ be a group and let $a \in G$. Then \{ $a^n$ | $n \in \mathbb{Z}$ \} is a subgroup of called the cyclic subgroup of $a$, denoted $\langle a \rangle$.

ex.: in $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$, op. is $+_6$
\[ \langle 2 \rangle = \{0, 2, 4\} = \langle 4 \rangle; \quad \langle 3 \rangle = \{0, 3\} \]
\[ \langle 5 \rangle = \{0, 5, 4, 3, 2, 1\} = \langle 1 \rangle \]

ex.: in $\mathbb{Z}$; \[ \langle 2 \rangle = \{\ldots -4, -2, 0, 2, 4, \ldots \} = \mathbb{Z} \]

Defn: An element $a$ of a group $G$ generates $G$ if $\langle a \rangle = G$. A group is called cyclic if there is some $a \in G$ that generates $G$.

ex: $\mathbb{Z}_6$ is cycle with generators 1 and 5.

Defn: Let $G$ be a group. Let $a \in G$. If the cyclic group $\langle a \rangle$ of $G$ is finite then the order of $a$ is the number of elements in $\langle a \rangle$, that is $|\langle a \rangle|$. If $\langle a \rangle$ does not have finite order then the order of $a$ is said to be infinite.

exs: $2 \in \mathbb{Z}_6$ has order 3
$2 \in \mathbb{Z}$ has infinite order
$2 \in \mathbb{Z}_5$? \[ \langle 2 \rangle = \{0, 2, 4, 1, 3\} \]
So the order of 2 is 5 and 2 generates $\mathbb{Z}_5$.
Division Algorithm for \( \mathbb{Z} \): If \( m \) is a positive integer and \( n \) is any integer then there exist unique integers \( q \) and \( r \) such that
\[
n = mq + r \quad \text{with} \quad 0 \leq r < m
\]

Remark: "usual proof" relies on the well ordering principle (every non-empty set of positive integers contains a smallest element).

ex: In \( \mathbb{Z}_6 \) find \( 3 \cdot (2) = 2 + 2 + 2 = 0 \)
\[\text{add 2 to itself 3 times where add is +}_6\]

find \( 101 \cdot (2) = [33 \cdot (3) + 2] \cdot (2) = 4 \) +_6 of 2, 101 times

Theorem: Every cyclic group is abelian.