GP - GEB Problem Set: Propositional Calculus

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Abstract: This problem set is based on Chapter 7 of Hofstadter's GEB. The problem set features a rather idiosyncratic presentation of the propositional calculus. That said, the presentation nicely contextualizes the propositional calculus within the realms of human reasoning and mathematical reasoning.

The Questions

1. Write down the nine shortest atoms in Hofstadter's presentation of the propositional calculus.

   P, P',P'', Q, Q', Q'', R, R', R''

2. Thinking of the propositional calculus in the terms that Hofstadter presents it, that is, as the formal system he constructs in the chapter:

   a. How many axioms in the formal system?

      None, only rules.

   b. How many rules in the formal system?

      9

   c. What are the names that he gives to these rules?

      Joining, Separation, Double-Tilde, Fantasy, Carry-over, Detachment, Contrapositive, De Morgan's, Switcharoo

   d. What is the one rule that you absolutely must use if you are to derive a theorem in this system?

      Fantasy Rule

3. Write down each of the rules of the system, just as Hofstadter does on page 187.

   <u>Joining Rule:</u> If *x* and *y* are theorems, then < *x* ∧ *y* > is a theorem

   <u>Separation Rule:</u> If < *x* ∧ *y*> is a theorem, then both *x* and *y* are theorems.

Double-Tilde Rule: **The string '~~' can be deleted from any theorem. It can also be inserted into any theorem, provided that the resulting string is itself well-formed.**

Fantasy Rule: **If *y* can be derived when *x* is assumed to be a theorem, then < *x* ⊃ *y* > is a theorem.**

Carry-Over Rule: **Inside a fantasy, any theorem from the "reality " one level higher can be brought in and used.**

Rule of Detachment: **If *x* and < *x* ⊃ *y* > are both theorems, then *y* is a theorem.**

Contrapositive Rule: **< *x* ⊃ *y* > and < ~*y*⊃~*x* > are interchangeable.**

De Morgan's Rule: **< ~*x*∧~*y* > and ~< *x* ∨ *y* > are interchangeable.**

Switcharoo Rule: **< *x* ∨ *y* > and <~ *x* ⊃*y* > are interchangeable.**

4. Derive: <<< P ∧ Q > ∧ R > ⊃ < P ∧ < Q ∧ R >>>

[	**push**
<< P ∧ Q > ∧ R >	**premise**
<P ∧ Q>	**separation**
P	**separation**
Q	**separation**
R	**separation**
<Q ∧ R>	**joining**
<P ∧ <Q ∧ R>	**joining**
]	**pop**
<<< P ∧ Q > ∧ R > ⊃ < P ∧ < Q ∧ R >>>	

5. Derive: < < P ∨ Q > ⊃ < Q ∨ P > >

[	**push**
~<P ∨ Q>	**premise**
<~P ∧~Q >	**demorgans**
~P	**separation**
~Q	**separation**
<~Q ∧ ~P>	**joining**
~<Q ∨ P>	**demorgans**
]	**pop**
<~<P ∨ Q> ⊃ ~<Q ∨ P>>	**fantasy**
< < P ∨ Q > ⊃ < Q ∨ P > >	**contrapositive**

6. Derive a theorem in the propositional calculus that you think is a little bit interesting, one that neither I asked you to derive, nor Hofstadter derived in his book.

Derive: **<<R ∨ Q> ⊃ <P ∨ Q >>**

[	**push**
~<R ∨ Q>	**premise**
<~R ∧ ~Q>	**demorgans**
~R	**separation**
~Q	**separation**
[	**push**
~P	**premise**
~Q	**carry-over**
<~P∧~Q>	**joining**
]	**pop**
<~P ⊃ <~P∧~Q>>	**fantasy**
<~P ⊃ ~< P ∨ Q>>	**demorgans**
~<P ∨ Q>	**detachment**
]	**pop**
<~<R ∨ Q> ⊃ ~<P ∨ Q >>	**fantasy**
<<R ∨ Q> ⊃ <P ∨ Q >>	**contrapositive**

7. As Hofstadter mentions mid-way through the chapter, there is a decision procedure for WFFs in the propositional calculus, the method of truth tables. Learn what this method entails, if you are not already clear on that, and write a description of the method that is clear and complete enough that one could easily apply it by referencing your description. That is, describe the process featuring truth tables by which one could determine whether or not a WFF is a theorem in the propositional calculus.

   The truth value of a WFF is determined by the truth values of its arguments. A table is created with a column for each proposition/argument, and every possible combination of truth-values are listed.

8. Using the truth table-based decision procedure, show that the heads will be cut off! Perhaps I should say a bit more. I'm referring to the section on Ganto's Ax. And I'm asking you to show by means of a truth table that the following WFF is a theorem: <<< P ⊃ Q > ∧ < ~ P ⊃ Q > > ⊃ Q >

P	Q	P⊃Q	~P⊃Q	<P ⊃ Q> ∧ <~P ⊃ Q>	<<< P ⊃ Q > ∧ < ~ P ⊃ Q > > ⊃ Q >
T	T	T	T	T	T
F	F	T	F	F	T
T	F	F	T	F	T
F	T	T	T	T	T

9. Choose another interpretation for P and Q in Ganto's statement, one that doesn't involve heads or axes. Write down the words for your proposition P. Write down the words for you proposition Q. Write down a sentence corresponding to Ganto's statement (what he says to the praying monks) under your interpretation.

   Q: If you walk outside

   P: You will be struck by the gods

   <<P⊃Q>∧<~P⊃Q>>

   If you walk outside, you will be struck by the gods, or, if you don't walk outside, you will be struck by the gods.

10. Write down in a meaningful manner, in no more than a few sentences, what you think is the most salient idea that Hofstadter has embedded in the text contained within the section titled *Shortcuts and Derived Rules*.

Derived rules are carried out while in I-mode rather than through a derivation within the system. However, these derived rules cannot be considered as explicit rules toward the system because of the informal way in which it was derived.

11. Write down in a meaningful manner, in no more than a few sentences, what you think is the most salient idea that Hofstadter has embedded in the text contained within the section titled, *Formalizing Higher Levels*.

    Fallacies result if we fail to distinguish between working in M and I modes. If a system is able to think about itself, it still isn't *outside* itself, and will be perceived differently by anything that isn't the system.

12. Write down in a meaningful manner, in no more than a few sentences, what you think is the most salient idea that Hofstadter has embedded in the text contained within the section titled, *Reflections on the Strengths and Weaknesses of the System*.

    Simplicity and precision are the main reasons why propositional calculus is appealing to mathematicians. Propositional Calculus can be easily extended to include other fundamental aspects of reasoning, and it can also be studied for its own properties (using different rules of inference, symbols, axioms, etc.)

13. Write down in a meaningful manner, in no more than a few sentences, what you think is the most salient idea that Hofstadter has embedded in the text contained within the section titled, *Proofs vs Derivations*.

    Derivations are the artificial, formal counterparts to the informal "proof". Propositional Calculus should be thought of as part of a general method for synthesizing artificial proof-like structures, but not with much flexibility or generality.

14. Write down in a meaningful manner, in no more than a few sentences, what you think is the most salient idea that Hofstadter has embedded in the text contained within the section titled, *The Handling of Contradictions*.

    If there is an incompleteness or inconsistency within a formal system, one can be sure that it is the fault of the larger system and not that of the subsystem, propositional calculus.

15. In one paragraph, write your reaction to this chapter.

This chapter was both incredibly informative, and incredibly confusing at the same time. I think the most intriguing part of the chapter was learning about the fantasy rule, and its recursive possibilities. Allowing for recursion within a formal system adds way more complexity which both scares/excites me.