Rochester Institute of Technology
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On the Classical Ramsey Number $R(3,3,3,3)$

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Abstract

The classical Ramsey Number $R(3,3,3,3)$, which is the smallest positive integer n such that any edge coloring with four colors of the complete graph on n vertices must contain at least one monochromatic triangle, is discussed. Basic facts and graph theoretic definitions are given. Papers concerning triangle-free colorings are presented in a historical overview. Mathematical theory underlying the main result of the thesis, which is Richard Kramers unpublished result, $R(3,3,3,3) \leq 62$, is given. The algorithms for the computational verification of this result are presented along with a discussion of the software tools that were utilized to obtain it.

Dedicated to:

Elizabeth G. Fettes

and

Robert C. Fettes

The best parents ever.

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Contents

Chapter 1

Introduction

In this thesis we discuss the classical Ramsey Number $R(3,3,3,3)$, which is the smallest integer n such that any edge coloring with four colors of the complete graph on n vertices K_n must contain at least one monochromatic triangle. Basic facts and graph theoretic definitions are given in Chapter 2.

In Chapter 3 we give a historical overview. Papers that have been written concerning triangle-free colorings are discussed. Different approaches for construction of triangle-free colorings are presented as are the arguments for nonexistence results which establish upper bounds. The history of triangle-free colorings is summarized.

In Chapter 4 we give the mathematical theory underlying the main result of this thesis, which is $R(3,3,3,3) \leq 62$. This upper bound was first established by Richard Kramer [13] in a series of talks at a graph theory seminar at Iowa State University in the spring of 1994. A computer-free proof appears in his unpublished manuscript. In Chapter 5 we give the algorithms for the computational verification of $R(3,3,3,3) \leq 62$. The software tools that we utilized are discussed in Chapter 6. Chapter 7 contains some final thoughts.

Chapter 2

Notation and Basic Concepts

A *graph* G is a finite nonempty set of objects V called *vertices* along with a (possibly empty) set E of unordered pairs of distinct vertices of G called *edges.* The vertex set of G is denoted $V(G)$, while the edge set is denoted $E(G)$. So $G = (V, E) = (V(G), E(G))$. If $\{u, v\} \in E(G)$ then we say that u and v are *adjacent vertices.*

We can describe a graph G using an *adjacency matrix* . Suppose $V(G) = \{v_1, v_2, \ldots, v_n\}$. The n by n adjacency matrix $A(G) = [a_{ij}]$ is given by:

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in E(G) \\ 0 & \text{if } \{v_i, v_j\} \notin E(G) \end{cases}$$

Such an adjacency matrix will be a symmetric matrix containing entries that are either 0 or 1, having 0 entries along the main diagonal.

Two graphs can be "essentially the same" in that they can be represented by identical diagrams. We make this idea exact by defining the notion of *graph isomorphism.* A graph G is *isomorphic* to H if there exists a one-to-one onto mapping $\phi : V(G) \to V(H)$ that preserves adjacency. That is, $\{u, v\} \in E(G)$ if and only if $\{\phi(u), \phi(v)\} \in E(H)$. We see that "is isomorphic to" is an equivalence relation on the set of all graphs so if G is isomorphic to H we can say that G and H are isomorphic and we write $G \cong H$. A graph is *complete* if every two of its vertices are adjacent. There is just one complete graph on n vertices, up to isomorphism. We denote it K_n.

A *bipartite graph* G is a graph whose vertex set can be partitioned into two subsets V_1 and V_2 such that each edge in G has one end in V_1 and the other in V_2. If each vertex in V_1 is adjacent to each vertex in V_2 then G is a *complete bipartite graph,* and if $|V_1| = m$ and $|V_2| = n$, we write $G = K_{m,n}$.

An $(r_1, r_2, \ldots, r_k)$-*coloring,* $r_i \geq 1$ for $1 \leq i \leq k$, is an assignment of one of k colors to each edge in a complete graph, such that it does not contain any monochromatic complete subgraphs K_{r_i} in color i, for $1 \leq i \leq k$. Also, an $(r_1,\, r_2,\, \ldots,\, r_k; n)$-*coloring* is an $(r_1,\, r_2,\, \ldots,\, r_k)$-coloring of K_n. The Ramsey number $R(r_1, r_2, \ldots, r_k)$ is defined to be the least $n > 0$ such that the set of all $(r_1,\, r_2,\, \ldots,\, r_k; n)$-colorings is empty.

Two k-colorings are *isomorphic* if there exists a one-to-one onto mapping between the vertices of the underlying complete graphs preserving all the colors of the edges, and they are *weakly isomorphic* if there exists a bijection between vertices that preserves the relation of two edges having the same color. In our work to construct colorings, the assignments of one of k colors to each edge in a complete graph may be only partial. In this case, we consider any edge that has not yet been assigned a color to have color 0, and call such a coloring a *partial coloring.* Each partial coloring can then be considered as a $(k+1)$-coloring with the extra color being color 0. It then makes sense to talk about partial colorings being isomorphic.

In this thesis the main Ramsey number under consideration is $R(\underbrace{3, 3, \ldots, 3}_{k\ times}) = R_k(3)$ which is the smallest integer n such that any edge coloring with k colors of the complete graph on n vertices must contain at least one monochromatic triangle. We will call an edge coloring of K_n *good* if no monochromatic triangles are formed. Then, for instance, the Ramsey number $R(3, 3, 3, 3)$ is the smallest n for which there is no good 4-coloring of K_n.

Let V be the vertex set of an edge-colored complete graph. Let α be a color. For $v \in V$, define the neighborhood of v of color α, denoted $N_\alpha(v)$, to be the set of vertices whose edges to v are of color α. We refer to $|N_\alpha(v)|$ as the *degree of* v *in color* α and denote it by $deg_\alpha(v)$. Now, for u and v two distinct vertices and δ any color, the set $N_\delta(u) \cap N_\delta(v)$ is referred to as a *u-v attaching set,* or just an *attaching set,* if u, v and δ are clear from the context.

If u and v are vertices in an edge-colored graph and α is a color, we write $u \overset{\alpha}{\text{———}} v$ to indicate that the edge connecting u and v has color α.

Define $[i_1, \ldots, i_n] = \{\ (i_{f(1)}, \ldots, i_{f(n)}) \mid f \text{ is a permutation of } \{1, \ldots, n\}\ \}$.

Suppose K_n has a good edge coloring in colors α, β, γ, and δ. Then for any $\eta \in \{\alpha, \beta, \gamma, \delta\}$ and for any $v \in V$ the induced edge coloring on the complete graph with vertex set $N_\eta(v)$ cannot contain any edges of color η. That is, $N_\eta(v)$ inherits a good 3-coloring. Thus, the order of each $N_\eta(v)$ must be less than $R(3,3,3) = 17$ [10]. Therefore, $(|N_\alpha(v)|, |N_\beta(v)|, |N_\gamma(v)|, |N_\delta(v)|) \in [a,b,c,d]$ where a, b, c, d are nonnegative integers that sum to $n-1$. We refer to $[deg_\alpha(v), deg_\beta(v), deg_\gamma(v), deg_\delta(v)]$ as a *color degree sequence* for a $(3,3,3,3;n)$-coloring. The possibilities for the color degree sequences of a fixed vertex in V for $n = |V|$, $59 \le n \le 65$, are given in Table 2.1.

n	possible orders of $N_\eta(v)$
65	[16, 16, 16, 16]
64	[16, 16, 16, 15]
63	[16, 16, 16, 14]
	[16, 16, 15, 15]
62	[16, 16, 16, 13]
	[16, 16, 15, 14]
	[16, 15, 15, 15]
61	[16, 16, 16, 12]
	[16, 16, 15, 13]
	[16, 16, 14, 14]
	[16, 15, 15, 14]
	[15, 15, 15, 15]
60	[16, 16, 16, 11]
	[16, 16, 15, 12]
	[16, 16, 14, 13]
	[16, 15, 15, 13]
	[16, 15, 14, 14]
	[15, 15, 15, 14]
59	[16, 16, 16, 10]
	[16, 16, 15, 11]
	[16, 16, 14, 12]
	[16, 16, 13, 13]
	[16, 15, 15, 12]
	[16, 15, 14, 13]
	[15, 15, 15, 13]
	[15, 15, 14, 14]

Table 2.1: Color degree sequences for $(3,3,3,3,\ge 59)$-colorings.

All good 3-colorings of K_{15} and K_{16} are known (see Chapter 3). So, when a certain neighborhood in a good 4-coloring has order at least 15, the possible colorings are limited to two good 3-colorings of K_{15} and two good 3-colorings of K_{16}. Note that for $n = 64$ all four neighborhoods have order at least 15. The proof that $R(3,3,3,3) \le 64$ [20] uses this fact. For $n \ge 62$

for at least three out of four colors, the neighborhoods must have order at least 15. This is the basis for our approach to show that $R(3,3,3,3) \leq 62$. For $n \geq 60$ for at least two out of four colors, the neighborhoods must have order at least 15. It is possible that this might be used to further lower the upper bound on $R(3,3,3,3)$ to 60.

Chapter 3

Historical Overview

In this section we discuss the papers that have been written concerning triangle-free colorings of complete graphs in the context of Ramsey Theory, along with some basic results about triangle-free colorings.

Consider first $R(3,3)$, the smallest integer n such that any edge-coloring with two colors of the complete graph on n vertices must contain at least one monochromatic triangle.

<u>Theorem 3.1</u>: $R(3,3) = 6$.

<u>Proof</u>:
We first demonstrate that K_5 has a good edge-coloring in two colors. Color a pentagon one color and inscribe a star of a different color within. That is, let V be the vertex set $\{\ v_0,\ v_1,\ v_2,\ v_3,\ v_4\ \}$. Let α, β be colors. Consider the coloring given by $v_0 \overset{\alpha}{\text{——}} v_1 \overset{\alpha}{\text{——}} v_2 \overset{\alpha}{\text{——}} v_3 \overset{\alpha}{\text{——}} v_4 \overset{\alpha}{\text{——}} v_0$ and $v_0 \overset{\beta}{\text{——}} v_2 \overset{\beta}{\text{——}} v_4 \overset{\beta}{\text{——}} v_1 \overset{\beta}{\text{——}} v_3 \overset{\beta}{\text{——}} v_0$. It is easy to see that this is a good 2-coloring of K_5.

Next, suppose K_n has a good coloring in two colors α and β. For v in the vertex set V, consider $\{v\},\ N_\alpha(v),\ N_\beta(v)$, which form a partition of V. We easily see that $|N_\alpha(v)| \leq 2$ and $|N_\beta(v)| \leq 2$. Thus,

$$\begin{aligned} n &= |V| \\ &= |\{v\}| + |N_\alpha(v)| + |N_\beta(v)| \\ &\leq 1 + 2 + 2 = 5 \end{aligned}$$

We have given a good 2-coloring on K_5 and shown that if K_n has a good

2-coloring then $n \leq 5$, therefore $R(3,3) = 6$. □

Interestingly, the problem of finding $R(3,3)$ was posed in a 1955 article by R. E. Greenwood and A. M. Gleason [10] as it appeared as a question in the March 1953 Putnam exam:

> Six points are in general position in space (no three in a line, no four in a plane). The fifteen line segments joining them in pairs are drawn, and then painted, some segments red, some blue. Prove that some triangle has all its sides the same color.

In this Greenwood and Gleason article, they show $R(3,3) = 6$, give the first proof that $R(3,3,3) = 17$, and show $42 \leq R(3,3,3,3) \leq 66$, making this the first paper to survey triangle-free colorings.

The proof that $R(3,3,3) = 17$ consists of two parts. $R(3,3,3) \geq 17$ was shown by giving a good 3-coloring of K_{16}. The construction relies on finite field theory using the Galois Field of order 16 and considers elements of the field to be the vertices of a graph. The cubic residues in the multiplicative group of the non-zero field elements are given. There are five cubic residues giving rise to three cosets. An edge is then colored according to which coset the difference of its vertices belongs. It is then shown that such a coloring contains no monochromatic triangles.

The authors go on to show that any 3-coloring of K_{17} must contain a monochromatic triangle by considering the possible orders of the neighborhoods of a fixed vertex, much as we did above for $R(3,3)$. The same argument works to show $R(3,3,3,3) \leq 66$, and we will present it when we focus our attention on $R(3,3,3,3)$. Thus, we have:

Theorem 3.2 [10]: $R(3,3,3) = 17$.

The next major item in the literature of triangle-free colorings is the 1968 article by J. G. Kalbfleisch and R. G. Stanton [12]. There are two good 3-colorings of K_{16} and both were known before the article. The first is the one described above. The discovery of the second good 3-coloring of K_{16} is attributed by Kalbfleisch and Stanton to Mr. Lee Jones, who wrote a search program using computers at the University of Waterloo during the summer of 1966, while he was an undergraduate. It is in this article by Kalbfleisch and Stanton that the second coloring makes its first appearance in the literature.

Kalbfleisch and Stanton also gave a new construction which yielded both (3,3,3;16) colorings. The constructions did not rely on finite field theory but instead on an argument that in a good 3-coloring of K_{16} the subgraph formed by the 16 vertices and the edges of any one color is isomorphic to a given graph. Then, any good 3-coloring of K_{16} may be obtained by appropriately fitting together three copies of this graph. It is shown that there are only two ways in which that can be done. So, Kalbfleisch and Stanton prove the following theorem: There are exactly two non-isomorphic good 3-colorings on 16 vertices, and they are not weakly isomorphic to each other.

```
                        111112222233333  A
                       1 22331132211332  A
                       12 3231221332131  B
                       123 322123123311  B
                       1323 22312131123  C
                       13322 3211213213  C
                       211223 313132312  B
                       2121323 11323132  B
                       23221111 3333221  A
                       221321313 121233  D
                       2231121331 12323  D
                       31323132321 2121  C
                       312313233122 211  C
                       3313123122312 12  D
                       33312113232211 2  D
                       321133221331122   A

color degree sequence: 5555555555555555,  40 edges in color 1
                       5555555555555555,  40 edges in color 2
                       5555555555555555,  40 edges in color 3
```

Figure 3.1: Twisted good 3-coloring of K_{16}.

Incidence matrices are used to represent the colorings. An *incidence matrix* is an array $A = [a_{ij}]$, used to hold the colors of the edges. That is, $a_{ij} = c$ if $v_i \overset{c}{\text{———}} v_j$. Diagonal entries do not correspond to an edge and are omitted. Such arrays are necessarily symmetric. Beneath the incidence matrix we give the color degree sequence for each of the vertices (using one hexadecimal digit for each degree), written as a column, and the total number of edges of each color.

We give the incidence matrices for the two colorings in (3,3,3,3;16) in Figure 3.1 and Figure 3.2. The terms *twisted* and *untwisted* are explained in the discussion of the next article as are the other markings in the figures. We see that the color degree sequence for each vertexis $[5, 5, 5]$.

```
                        111112222233333  A B
                       1 22331132211332  A
                       12 3321221323131    B
                       123 232123132311
                       1332 23212131123
                       13232 2311213213
                       211232 313123312
                       2121233 11332132
                       23221111 3333221  A
                       221321313 121233
                       2231121331 12323     B
                       31233123321 2121     B
                       313213323122 211
                       3313123122312 12
                       33312113232211 2
                       321133221331122   A

 color degree sequence: 5555555555555555,  40 edges in color 1
                        5555555555555555,  40 edges in color 2
                        5555555555555555,  40 edges in color 3
```

Figure 3.2: Untwisted good 3-coloring of K_{16}.

Yet another construction of the two non-isomorphic good 3-colorings of K_{16} was given by C. Laywine and J. P. Mayberry [14] in their 1988 article. The approach is similar in spirit to Kalbfleisch and Stanton [12] in that finite field theory was not used. Instead, the good colorings were built. While Kalbfleisch and Stanton fitted together graphs of a single color, the building blocks of Laywine and Mayberry were good colorings of K_4 called tri-colored tetrahedrons (TCTs). A TCT was defined as follows. Let V be the vertex set $\{v_0,\, v_1,\, v_2,\, v_3\}$. Let $\alpha,\ \beta,\ \gamma$ be colors. Color the edges in V by: $v_0 \overset{\alpha}{\rule{2em}{0.4pt}} v_2$, $v_1 \overset{\alpha}{\rule{2em}{0.4pt}} v_3$, $v_1 \overset{\beta}{\rule{2em}{0.4pt}} v_2$, $v_0 \overset{\beta}{\rule{2em}{0.4pt}} v_3$, $v_0 \overset{\gamma}{\rule{2em}{0.4pt}} v_1$,

$v_2 \xrightarrow{\gamma} v_3$. That is, every vertex is incident with one edge of each color. A single TCT is given in Figure 3.3.

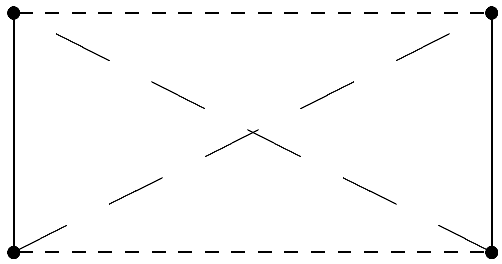

Figure 3.3: A tri-colored tetrahedron (TCT).

These TCT's were then fitted together to make each of the two good 3-colorings of K_{16}. One of them was called untwisted by the authors, and it is isomorphic to the construction done by Greenwood and Gleason [10]. The untwisted coloring has the property that every edge belongs to exactly one TCT. The other was called twisted and is isomorphic to the one found by computer search and given in the Kalfleisch and Stanton aritcle [12]. We denote the twisted coloring T_1 (Figure 3.1) and the untwisted coloring T_2 (Figure 3.2). T_1 contains only the four vertex disjoint TCT's that were used in its construction. So, every vertex belongs to a unique TCT in T_1. Thus, a good 3-coloring of K_{16} can be classified as being isomorphic to T_1 or T_2 by doing the following. Choose an arbitrary edge. If the edge does not belong to a TCT then the coloring is a isomorphic to T_1. If it does belong to a TCT then choose a second edge with the same color having a single vertex in common with the first edge. If this new edge also belongs to a TCT the coloring is isomorphic to T_2, otherwise it is isomorphic to T_1. The letter markings to the side of Figure 3.1 partition the vertices into four sets each of whose induced subcolorings is a TCT. The letter markings to the side of Figure 3.2 give overlapping sets of vertices of order four each of whose induced subcolorings is a TCT.

We now focus our attention on $R(3,3,3,3)$. As mentioned, Greenwood and Gleason [10] showed $R(3,3,3,3) \leq 66$. We fill in the details by giving the standard argument.

Theorem 3.3: $R(3,3,3,3) \leq 66$.

Proof:
Let K_n have a good 4-coloring in colors α, β, γ, δ. Let v be a fixed vertex in the vertex set V for K_n. Consider the induced coloring on $N_\eta(v)$ for each $\eta \in \{\alpha, \beta, \gamma, \delta\}$. $N_\eta(v)$ does not contain edges of color η otherwise there would be a triangle in color η, and hence $N_\eta(v)$ exhibits a good 3-coloring. Thus, since $R(3,3,3) = 17$, each $|N_\eta(v)| \leq 16$. Since $\{v\}$, $N_\alpha(v)$, $N_\beta(v)$, $N_\gamma(v)$, $N_\delta(v)$ form a partition of V, we have:

$$\begin{aligned} n &= |V| \\ &= |\{v\}| + |N_\alpha(v)| + |N_\beta(v)| + |N_\gamma(v)| + |N_\delta(v)| \\ &\leq 1 + 16 + 16 + 16 + 16 = 65 \end{aligned}$$

That is, if K_n has a good 4-coloring then $n \leq 65$. Therefore, $R(3,3,3,3) \leq 66$. $\square$

The result $R(3,3,3,3) \leq 65$ appeared first in a 1973 paper by E. Whitehead [24], although he gives credit for part of the proof to J. Folkman. Notes by Folkman were printed posthumously in 1974 [6]. The proof that $R(3,3,3,3) \leq 65$ uses linear algebra techniques for analyzing the structure of the two good 3-colorings on K_{16}. It is interesting to note that the author cites using an IBM 360/Model 65 program to verify certain properties of these colorings.

No progress was made on lowering the upper bound for $R(3,3,3,3)$ until A. Sánchez-Flores [20] gave a computer-free proof that $R(3,3,3,3) \leq 64$. In this 1995 article, Sánchez-Flores presented properties of the good 2-colorings of K_5, the good 3-colorings of K_{16}, as well as properties that a good 4-coloring of K_n must satisfy, which are then used to derive the new upper bound. Sánchez-Flores proves a key lemma which shows that certain attaching sets (namely those which contain a monochromatic $K_{1,5}$) are not possible in a good 4-coloring of K_n, unless $n \leq 60$. Here is his lemma 2.4, which has been reworded due to differences in notation.

Sánchez-Flores Lemma: Suppose K_n has a good 4-coloring in colors $\alpha, \beta, \gamma, \delta$ with the property that for each vertex v in the vertex set V, and for each color $\eta \in \{\alpha, \beta, \gamma, \delta\}$, the induced coloring on $N_\eta(v)$ is contained in a good 3-coloring of K_{16}. If there exist $x, y \in V$ and $\eta \in \{\alpha, \beta, \gamma, \delta\}$ such that

$|N_\eta(x)| = 16 = |N_\eta(y)|$ and $a_0, a_1, a_2, a_3, a_4, a_5 \in N_\eta(x) \cap N_\eta(y)$ along with $\mu \in \{\alpha, \beta, \gamma, \delta\} \backslash \eta$ such that $a_h \in N_\mu(a_0)$, for $h = 1, \ldots, 5$, then $N_\eta(x) = N_\eta(y)$ and $n \leq 60$.

As we noted previously, for $n = 64$, the possible color degree sequences for a fixed vertex show that each of the color neighborhoods must be of order at least 15 (see Table 2.1 and the discussion that went along with it). Further, K. Heinrich showed, in a 1977 article, that deleting one point from each of the good 3-colorings of K_{16} leads to exactly two nonisomorphic $(3,3,3;15)$-colorings [11] and that no other $(3,3,3;15)$-colorings exist. Hence the Sánchez-Flores lemma applies in the $n = 64$ case.

The proof that $R(3,3,3,3) \leq 64$ supposes that a good 4-coloring exists on K_{64} and goes on to show that a vertex can be added in such a way as to obtain a good 4-coloring on K_{65}. This cannot happen since no such coloring exists, as shown by Whitehead [24] and Folkman [6].

In the spring of 1994 Richard Kramer [13] gave a series of talks at a graph theory seminar at Iowa State University to show that $R(3,3,3,3) \leq 62$. These talks led to an unpublished manuscript of considerable length (116 pages). Kramer's arguments for his claimed improvement of the upper bound on $R(3,3,3,3)$ rely on a combination of local and global methods. Local arguments form the bulk of the manuscript. This manuscript was, in some sense, the impetus for this thesis.

Now we'll trace through the literature on lower bounds for $R(3,3,3,3)$. As mentioned, the 1955 paper by Greenwood and Gleason [10] established the lower bound $R(3,3,3,3) > 41$. The authors did so by giving a good 4-coloring of K_{41} having vertices labeled by elements from the finite field having 41 elements. The usual finite field theory approach of coloring edges according to which coset of quartic residues the difference of its vertices belongs was indicated, but no details were given.

S. W. Golomb and L. D. Baumert [8] showed $R(3,3,3,3) > 45$ in a seemingly unconnected article on backtrack programming. The authors used a digital backtrack program to show that the greatest integer q such that $\{1, 2, \ldots, q\}$ can be partitioned into four sum-free sets is 44. By a sum-free set we mean a set of positive integers S such that $x, y \in S$ imply $x + y \notin S$. Although it was not noted in the article, this gives a good 4-coloring of K_{45} by labeling the vertices $1, 2, \ldots, 45$, associating to each of the sum-free sets a color, and then coloring each edge according to which set the difference of its vertices belongs. Sum-free sets in this context are also discussed by H. Fredricksen in [7].

Whitehead [24] in his 1973 paper, which also contained the proof attributed to Folkman for $R(3,3,3,3) \leq 65$, established the improved lower bound $R(3,3,3,3) > 49$. The good 4-coloring of K_{49} was obtained by using the notion of sum-free sets. The nonzero elements of the finite abelian group Z_7 x Z_7 were partitioned into four symmetric sum-free sets. Then a good 4-coloring was given using the procedure described above. A general theorem, showing that any such coloring obtained in this way using sum-free sets is triangle free, was given. In a previous paper [22] Whitehead used this type of construction to obtain one of the good 3-colorings of K_{16}. Other work by Whitehead on Ramsey theory appears in [22], [23].

In her 1973 article, F. R. K. Chung took an incidence matrix for one of the two good 3-colorings of K_{16} and constructed from it the incidence matrix corresponding to a good 4-coloring of K_{50}, thereby establishing $R(3,3,3,3) > 50$, which is to date the best known lower bound. In this article she notes that $R(3,3,3,3) > 50$ was proved independently by G. J. Porter of the University of Pennsylvania. We give the incidence matrix for the Chung coloring in Figure 3.4. Also, many nonisomorphic good 4-colorings of K_{50} were obtained by S. Radziszowski.

We summarize the history of triangle-free colorings in Table 3.1.

year	reference	lower	upper
1955	Greenwood, Gleason [10]	42	66
1967	false rumors	[66]	
1971	Golomb, Baumert [8]	46	
1973	Whitehead [24]	50	65
1973	Chung [1], Porter	51	
1974	Folkman [6]		65
1995	Sánchez-Flores [20]		64
1995	Kramer [13]		62
2001	Fettes, Radziszowski	51	62

Table 3.1: Historical Overview

Finally, we have gathered basic inequalities concerning $R_k(3)$, listed below. Any lowering of the upper bound on $R(3,3,3,3)$ will result in improving upper bounds for $R_k(3)$, $k > 4$. Increasing the lower bound of $R(3,3,3,3)$ will improve (3).

(1) $R_k(3) \leq k(R_{k-1}(3) - 1) + 2, \quad k \geq 2$

(2) $R_k(3) \geq 3R_{k-1}(3) + R_{k-3}(3) - 3, \quad k \geq 4$

(3) $R_k(3) \leq k!\left(\frac{e-e^{-1}+3}{2}\right) + 1, \quad k \geq 4$

(4) $3.16 < \lim_{k\to\infty} R_k(3)^{\frac{1}{k}} \quad \textit{exists}$

Inequality (1) can easily be shown by generalizing the proof of our Theorem 3.3. Inequality (2) is given by F. R. K. Chung [1] as a generalization of the specific case when $k = 4$. When $k = 4$ the proof by construction gives the incidence matrix of her good 4-coloring of K_{50}. Inequality (3) was obtained from inequality (1) using analysis techniques, by Wan Honghui [21] and inequality (4) was obtained from inequality (2) also using analysis techniques, by C. M. Grinstead [2].

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Coloring on 50 vertices with 4 colors

 2222233333444443222221111144444211111333334444411
2 3344224332244323114422411224411233441143311144311
23 434233244324221341421124412421324341331443144111
234 43323423442221434112142144221342433134134411111
2434 3342324223424143114212422141434233413141134111
24433 432232432424411341221241241443324311314314111
322334 42424342312211434242414213113342414143413111
3232434 2243424312124143224142413131434211434143111
34332222 444433214112222344441123433111124444331111
332432424 23234411241242432121443314314142131344111
3342232442 2343411422124423214143341131441213434111
42434243432 323242414241412312124143414343123131111
423424344233 32242142414421131224134143441332311111
4424234233423 2344242142114213214414134133413213111
44423224343322 344421224141122314441311434331123111
432244332442233 41224411244221134311443314411332111
3222221111144444 3333311111444441333332222244444221
23114422411224413 114433411334413122443342233442221
213414211244124231 41431134413433214243223442343221
2143411214214422314 4113143144333241422324324433221
24143114212422143414 114313433143424122432343324221
244113412212412434411 41331341343442214233234234221
1221143424241421133114 4343414312332241434342432221
12124143224142411313414 334143412323424133424342221
141122223444411214113333 44441132422333314444223221
1124124243212144113413434 3131442234234341323244221
11422124423214141143313443 314142243323443132424221
424142414123121243414341413 13134342434242312323221
4214241442113122431434144311 1334324342443221233221
44242142114213214434314311431 314434324322432132221
444212241411223144431334141133 14443233424223312221
4122441124422113413344113443311 4233442234433221221
21111133333444441333332222244444 111112222244444331
123344114331144331224433422334421 22441142211442331
1324341331443141321424322344234312 4241221442141331
13424331341344113241422324324433124 422124124411331
143423341314113434241224323433241424 22412141124331
1443324311314314344221423323423414422 4211214214331
31133424141434132332241434342432211224 414142412331
313143421143414323234241334243422121424 11424142331
3433111124444331242233331444422324221111 4444221331
33143141421313442234234341323244221421414 121244331
334113144121343422433234431324242241121441 12424331
4143414343123131434243424231232341424142421 2121331
41341434413323114324342443221233412414244122 211331
441413413341321344343243224321324414124122412 12331
4441311434331123444323342422331244412114242211 2331
43114433144113324233442234433221421144221441122 331
1111111111111111222222222222222233333333333333333 41
11111111111111112222222222222222333333333333333341

color  ccccccccccccccccbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbgg, 288 edges in color 1
degree bbbbbbbbbbbbbbbbccccccccccccccccbbbbbbbbbbbbbbbbgg, 288 edges in color 2
seqs:  bbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbccccccccccccccccgg, 288 edges in color 3
       ffffffffffffffffffffffffffffffffffffffffffffffff11, 361 edges in color 4
```

Figure 3.4: Good 4-coloring of K_{50} by Chung

Chapter 4

Mathematical Theory

In this section we give the mathematical theory underlying the main result $R(3,3,3,3) \leq 62$. Lemma 4.1 below is essentially Kramer's Proposition 1 [13]. The proof of Theorem 4.2 is drawn from ideas similar to those contained in the same manuscript, although it doesn't appear there in its current form.

Lemma 4.1: If K_{62} has a good 4-coloring in colors α, β, γ, δ then for each v in the vertex set V,

$$(|N_\alpha(v)|, |N_\beta(v)|, |N_\gamma(v)|, |N_\delta(v)|)$$

$$\in [16,16,16,13] \cup [16,16,15,14] \cup [16,15,15,15].$$

Proof:
Suppose K_{62} has a good 4-coloring in colors $\alpha, \beta, \gamma, \delta$. Let $v \in V$. Since $\{v\}$, $N_\alpha(v)$, $N_\beta(v)$, $N_\gamma(v)$, $N_\delta(v)$ form a partition of V we have,

$$\begin{aligned} 62 &= |V| \\ &= 1 + |N_\alpha(v)| + |N_\beta(v)| + |N_\gamma(v)| + |N_\delta(v)| \end{aligned}$$

So, $61 = |N_\alpha(v)| + |N_\beta(v)| + |N_\gamma(v)| + |N_\delta(v)|$. Moreover, for each $\eta \in \{\alpha, \beta, \gamma, \delta\}$, the induced coloring on $N_\eta(v)$ must contain no edges of color η, otherwise there would be a triangle of color η in the original coloring. Thus, $N_\eta(v)$ exhibits a good 3-coloring and hence $|N_\eta(v)| \leq 16$. The only partitions of 61 into four nonnegative integers each at most 16 are given, and so the lemma follows. $\square$

We note the importance of the partitions of $n-1$ into four integers each at most 16 when considering a good 4-coloring of K_n. It is precisely these color degree sequences that allow for the arguments contained in this thesis. We discussed this further and considered color degree sequences for other n's in Chapter 2 (see Table 2.1). Lemma 4.1 appears as part of this table.

Theorem 4.2: If K_{62} has a good 4-coloring in colors α, β, γ, δ then there exist vertices u, v along with a color (without loss of generality, δ) and a k, $3 \leq k \leq 14$, such that the u-v attaching set $N_\delta(u) \cap N_\delta(v)$ has order k. Moreover, $|N_\delta(u)| = 16 = |N_\delta(v)|$.

Proof:
Let V be the vertex set of a K_{62} with a good 4-coloring in colors $\alpha, \beta, \gamma, \delta$. By lemma 4.1, for each $v \in V$ there exists an $\eta \in C = \{\alpha, \beta, \gamma, \delta\}$ such that $|N_\eta(v)| = 16$. So for

$$X = \{\ (v, \eta) \mid |N_\eta(v)| = 16, v \in V, \eta \in C\ \}$$

we have $|X| \geq 62$. For $\eta \in C$, define $V_\eta = \{v \mid (v, \eta) \in X\}$. Then $V = V_\alpha \cup V_\beta \cup V_\gamma \cup V_\delta$ and $|V| = 62$ together imply that at least one of $V_\alpha, V_\beta, V_\gamma, V_\delta$ must have order ≥ 16. Without loss of generality, let δ be a color such that $|V_\delta| \geq 16$.

Let $z_0, z_1, z_2, z_3, z_4, z_5 \in V_\delta$ be distinct. We first show $|N_\delta(z_i) \cap N_\delta(z_j)| \geq 3$ for some distinct $i, j \in \{0, 1, 2, 3, 4, 5\}$. Suppose not, that is, $|N_\delta(z_i) \cap N_\delta(z_j)| \leq 2$ for all distinct $i, j \in \{0, 1, 2, 3, 4, 5\}$. Then,

$$\begin{aligned} 62 &= |V| \\ &\geq |N_\delta(z_0) \cup N_\delta(z_1) \cup N_\delta(z_2) \cup N_\delta(z_3) \cup N_\delta(z_4) \cup N_\delta(z_5)| \\ &\geq 16 + 14 + 12 + 10 + 8 + 6 \\ &= 66 \end{aligned}$$

leads to a contradiction. Thus, for some distinct $i, j \in \{0, 1, 2, 3, 4, 5\}$ we have $|N_\delta(z_i) \cap N_\delta(z_j)| \geq 3$. Let $u = z_i$ and $v = z_j$.

Now we will show that $|N_\delta(u) \cap N_\delta(v)| \leq 14$. Since $N_\delta(u) \cap N_\delta(v) \neq \emptyset$, the edge between u and v must be colored by one of α, β, γ. Without loss of generality, suppose $u \xrightarrow{\gamma} v$. Then, no vertex can have edges to both u and v colored by γ. That is, $N_\gamma(u) \cap N_\gamma(v) = \emptyset$. So, the sets $\{v\}, N_\gamma(u) \cap N_\alpha(v), N_\gamma(u) \cap N_\beta(v)$, and $N_\gamma(u) \cap N_\delta(v)$ form a partition of

$N_\gamma(u)$. Now, $N_\gamma(u) \cap N_\alpha(v)$ inherits a good 2-coloring in colors β, δ so $|N_\gamma(u) \cap N_\alpha(v)| \leq 5$. Similarly, $|N_\gamma(u) \cap N_\beta(v)| \leq 5$. Note also that lemma 4.1 implies $|N_\gamma(u)| \geq 13$. So,

$$\begin{aligned}13 &\leq |N_\gamma(u)| \\ &= |\{v\}| + |N_\gamma(u) \cap N_\alpha(v)| + |N_\gamma(u) \cap N_\beta(v)| + |N_\gamma(u) \cap N_\delta(v)| \\ &\leq 1 + 5 + 5 + |N_\gamma(u) \cap N_\delta(v)|.\end{aligned}$$

Thus, $|N_\gamma(u) \cap N_\delta(v)| \geq 2$.

Now consider the partition of the set $N_\delta(v)$ into $N_\alpha(u) \cap N_\delta(v), N_\beta(u) \cap N_\delta(v), N_\gamma(u) \cap N_\delta(v), N_\delta(u) \cap N_\delta(v)$. So we have,

$$\begin{aligned}16 &= |N_\delta(v)| \\ &\geq |N_\gamma(u) \cap N_\delta(v)| + |N_\delta(u) \cap N_\delta(v)| \\ &\geq 2 + |N_\delta(u) \cap N_\delta(v)|.\end{aligned}$$

Thus, $16 \geq 2 + |N_\delta(u) \cap N_\delta(v)|$, which implies $|N_\delta(u) \cap N_\delta(v)| \leq 14$. $\square$

Chapter 5

Algorithms and Computations

Suppose K_{62} has a good 4-coloring C in colors 1, 2, 3, 4. That is, let $C \in (3,3,3,3;62)$. Then, by Theorem 4.2, there are two distinct vertices u, v in the vertex set V and a color, which without loss of generality we can choose to be 4, such that the attaching set $N_4(u) \cap N_4(v)$ has order k, where $3 \leq k \leq 14$, and $|N_4(u)| = 16 = |N_4(v)|$. The main result in this thesis is to establish that no such attaching set can exist. Note that throughout this section all computational results were obtained independently by Stanisław Radziszowski and myself, compared, and no discrepancies were found.

The preliminary step is to identify all possibilities for induced colorings of attaching sets $N_4(u) \cap N_4(v)$. Such a set is a subset of $N_4(u)$ which, since $|N_4(u)| = 16$, has a coloring induced by one of the two good 3-colorings, T_1, T_2, of K_{16} presented in Chapter 3. We determine all possibilities for such attaching sets.

Proposition 5.1: There exist 533 nonisomorphic ways for a nonempty set of vertices to have a coloring induced on it by a good 3-coloring of K_{16}.

Proof: The following algorithm was executed for each of T_1, T_2. For each possible order k, $k = 1, 2, \ldots, 16$ (although $k = 3, 4, \ldots, 14$ suffices for our work by reasoning given in Chapter 4), for each nonempty subset S of vertices of K_{16} having order k, construct a partial coloring of K_{17} by adding a 17^{th} vertex and coloring the edge between the new vertex and each vertex in the set S with color 4. Pass the output through *shortmc* (see Chapter 6) to eliminate isomorphic copies. 533 partially colored K_{16}'s resulted. □

We say these partial colorings have *marked subsets* and corresponding *induced marked subcolorings.* See Table 5.1 for the results by order of the marked subset. Let Υ_1 denote this set of induced marked subcolorings of K_{16}. Agreement was reached in independent computations of Υ_1 as well as in the results for K_{15} summarized in Table 5.1.

During this phase a further lowering of the upper bound for $R(3,3,3,3)$ was still under consideration. Investigation of the possible color degree sequences for orders of each of the colored neighborhoods of a fixed vertex is given in Table 2.1, and there it was noted that for $n \geq 60$ for at least two out of four colors, the neighborhoods must have order at least 15. So, we also found nonisomorphic marked subsets in good 3-colorings of a K_{15}. There were 3402 of them. Note the symmetry of the results in Table 5.1 since a marked subset can be obtained from its complement.

order of marked subset of K_{16}	number	order of marked subset of K_{15}	number
1	2	1	3
2	3	2	13
3	7	3	51
4	21	4	145
5	34	5	312
6	66	6	517
7	83	7	659
8	99	8	659
9	83	9	517
10	66	10	312
11	34	11	145
12	21	12	51
13	7	13	13
14	3	14	3
15	2	15	2
16	2		
total	533	total	3402

Table 5.1: Nonisomorphic marked subsets in K_{15} and K_{16}.

Example 5.1: An example of an incidence matrix of a marked subset of order 5 within a good 3-coloring of a K_{16} is given in Figure 5.1. The color degree sequences for each vertex are given in hexadecimal below the matrix. For instance, (beginning the labeling of vertices with number 1), vertex 17 has five edges of color 4 attached to vertices numbered $1, 2, 3, 4, 5$. These vertices are the marked subset and correspond to vertices in the attaching set $N_4(u) \cap N_4(v)$. Vertex 17 corresponds to one of the vertices u or v. It is unnecessary to include a vertex corresponding to the other of u or v as the edges between it and all the other vertices (except the other of u or v) would have color 4.

```
         Coloring on 17 vertices with 4 colors

                  1111122222333334
                 1 223311322113324
                 12 32312213321314
                 123 3221231233114
                 1323 223121311234
                 13322 32112132130
                 211223 3131323120
                 2121323 113231320
                 23221111 33332210
                 221321313 1212330
                 2231121331 123230
                 31323132321 21210
                 312313233122 2110
                 3313123122312 120
                 33312113232211 20
                 321133221331122 0
                 4444400000000000

          color  00000111111111111b,  11 edges in color 0
         degree  55555555555555550,  40 edges in color 1
           seq:  55555555555555550,  40 edges in color 2
                 55555555555555550,  40 edges in color 3
                 11111000000000005,   5 edges in color 4
```

Figure 5.1: Marked subset of order 5 in T_1

Now we want to see how each of the marked subsets along with its induced marked subcoloring can be embedded in another good 3-coloring of K_{16}. By an embedding of the marked subset S, we mean here an injection $\phi : S \to V(T_i)$ such that for every $x_1 \neq x_2 \in S$ if $x_1 \stackrel{\eta}{-\!\!-} x_2$ then

$\phi(x_1) \overset{\eta}{\text{——}} \phi(x_2)$ for $\eta \in \{1, 2, 3\}$. That is, we want to construct all possible partial colorings of $N_4(u) \cup N_4(v)$ agreeing on the induced marked subcoloring of S. Each such partial coloring on $N_4(u) \cup N_4(v)$ can be considered as an overlapping of two good 3-colorings of K_{16}.

Proposition 5.2: There exist 724 nonisomorphic ways for two good 3-colorings of K_{16} to overlap.

Proof: The following algorithm was executed. For each partial coloring in Υ_1 embed the marked subset S in all possible ways into each of T_1, T_2. Using such an embedding, construct a partial coloring of a K_s ($s = 16 + 16 - k$, where k is the order of the marked subset S). Then pass the output through *shortmc* to eliminate isomorphic copies. 724 partial colorings resulted. □

See Table 5.2 for results by order of the marked subset.

file	number	file	number	file	number
15.15.1	17	15.16.1	6	16.16.1	3
15.15.2	268	15.16.2	50	16.16.2	7
15.15.3	1720	15.16.3	274	16.16.3	20
15.15.4	4369	15.16.4	725	16.16.4	54
15.15.5	5969	15.16.5	1066	16.16.5	74
15.15.6	6080	15.16.6	1178	16.16.6	109
15.15.7	5086	15.16.7	1071	16.16.7	110
15.15.8	3740	15.16.8	859	16.16.8	116
15.15.9	2283	15.16.9	590	16.16.9	91
15.15.10	1151	15.16.10	332	16.16.10	69
15.15.11	434	15.16.11	149	16.16.11	35
15.15.12	132	15.16.12	53	16.16.12	22
15.15.13	26	15.16.13	13	16.16.13	7
15.15.14	6	15.16.14	3	16.16.14	3
15.15.15	2	15.16.15	2	16.16.15	2
				16.16.16	2
total	31283	total	6371	total	724
grand total	38378				

Table 5.2: Overlapping colorings of K_{15} and K_{16}.

We only need the pairs of K_{16}'s case (the rightmost column of Table 5.2) but the other computations were done and verified so we also include those results. Additional cases will be needed if a similar argument can be made to further lower the upper bound on $R(3, 3, 3, 3)$. File $x.y.k$.mc contains colorings resulting from embedding a marked subset of order k in a good 3-coloring of K_x into a good 3-colorings on K_y. Note that $x.y.k$.mc must be the same, up to isomorphism, as $y.x.k$.mc, (a test that was passed successfully) so only 15.16.k.mc files are recorded. Then, except for 15.16.15.mc being equivalent to 16.16.16.mc, all others must be nonisomorphic. Thus, the total number of colorings is 38378, two more than nonisomorphic ones. Of these, 724 colorings are obtained by overlapping two good 3-colorings on K_{16} and it is these results that form the input for later phases of our work.

Example 5.2: A sample of a coloring from 16.16.5.mc is given in Figure 5.2. The x next to rows and columns $23, 24, 25, 26, 27$ indicates that these vertices are in the marked subset as they are precisely those vertices whose edges are fully colored (i.e., those which have no edges of color 0).

Let Υ_2 denote the set containing the 724 partial colorings from Proposition 5.2. Agreement was reached in independent computations of Υ_2 as well as the other results summarized in Table 5.2. Each of the objects in Υ_2 is a partial coloring of a graph with vertex set $N_4(u) \cup N_4(v)$. In order for one of these partial colorings to be contained in a full good 4-coloring of K_{62} we consider possible color degree sequences (see Table 2.1). At least two of the three degrees (for colors $1, 2, 3$) are at least 15, and each good 3-coloring of K_{15} is contained in one of the good 3-colorings of K_{16} [11]. It follows that for each vertex in the attaching set $N_4(u) \cap N_4(v)$, for at least two of the three colors {1,2,3}, the neighborhood in that color must be embeddable into one of the good 3-colorings K_{16}. By applying this restriction the number of partial colorings reduced from 724 to 129. Of these 129, we eliminated the five with attaching set of order 1 or 16 due to Theorem 4.2. This is still more than needed to show $R(3, 3, 3, 3) \leq 62$, as orders 2 and 15 will also not be used, again due to Theorem 4.2. However, inclusion provided additional correctness checks between the two implementations. The vertices u and v were added to the vertex set and the appropriate edges were colored with color 4. Let Υ_3 denote the set containing the 124 partial colorings obtained in this manner. See Table 5.3 for a breakdown of these partial colorings by order. Note the already all attaching sets of orders $8, 9, 10$ and 13 have been eliminated.

Example 5.3 : A sample of a coloring from Υ_3 is given in Figure 5.3. As in Example 5.2 the x next to rows and columns numbered $12, 13, 14, 15, 16$ indicates that these vertices correspond to those in the attaching set. Vertices numbered $28, 29$ correspond to u and v. The vertices in the attaching set

```
   Coloring on 27 vertices with 3 colors.
  Vertices in attaching set marked with x.

                              xxxxx

         00011002233000111003022233
        0 11002200003310001103222333
        01 300310000312000320122123
        013 00130000132000230121232
        1000 3003113000232001022123
        10003 001331000223001021232
        023100 10000232000120333112
        0213001 0000322000210331321
        20003100 132000221003033121
        200013001 23000212003031312
        3000130032 1000321002021321
        30003100231 000312002023112
        033100230000 13000210223121
        0313003200001 3000120221312
        01220022000033 000330231111
        100022002233000 33002031111
        1000320021210003 2003012331
        10002300121200032 003013213
        013200120000213000 20312331
        0123002100001230002 0313213
        30001100332200023300 012211
        031100330000222000330 12211
        2222223333222233111111 1133     x
        22212131311331112323221 332     x
        221212131331131132322213 23     x
        3323231221212111313111332 2     x
        33323221121212111313113232      x

 color  bbbbbbbbbbbbbbbbbbbbbb00000, 121 edges in color 0
degree  555555555555555555555899aa,  78 edges in color 1
   seq: 5555555555555555555555a9988,  77 edges in color 2
        5555555555555555555555588888,  75 edges in color 3
```

Figure 5.2: Partial coloring with vertex set $N_\delta(u) \cup N_\delta(v)$ from Υ_2.

number of vertices	order of attaching set	number of partial colorings
19	15	2
20	14	3
22	12	3
23	11	1
27	7	8
28	6	16
29	5	43
30	4	21
31	3	20
32	2	7
	total	124

Table 5.3: Partial colorings with vertex set $N_\delta(u) \cup N_\delta(v) \cup \{u, v\}, \Upsilon_3$.

are precisely those that have edges to both u and v of color 4.

We will now color additional edges in partial colorings with vertex set $N_4(u) \cup N_4(v) \cup \{u, v\}$ from Υ_3. We do so by using *embeddings* of the induced coloring of $N_c(x)$, for $c \in \{1, 2, 3\}$, into good 3-colorings of K_{16} in colors $\{1, 2, 3, 4\} \backslash \{c\}$, where x is the first vertex in the attaching set. For $i = 1, 2$ and $c \in \{1, 2, 3, 4\}$ let $T_i(c)$ denote the good 3-coloring of K_{16} in colors $\{1, 2, 3, 4\} \backslash \{c\}$ obtained by replacing color c in T_i by color 4 if $c \in \{1, 2, 3\}$ and by letting $T_i(4) = T_i$. We extend the previous definition of embedding to include handling color 0. By an embedding, we mean here an injection $\phi : N_c(x) \to V(T_i(c))$ such that for every $x_1 \neq x_2$ in the induced coloring of $N_c(x)$ if $x_1 \overset{\eta}{\text{———}} x_2$ then $\phi(x_1) \overset{\eta}{\text{———}} \phi(x_2)$ for $\eta \in \{1, 2, 3, 4\} \backslash \{c\}$. If the edge between x_1 and x_2 is uncolored then the edge between $\phi(x_1)$ and $\phi(x_2)$ can be of any color. Let C be a partial coloring. We say that we *pull back* an embedding, or *pull back* an embedding onto C, if for every $x_1 \neq x_2 \in N_c(x)$ with the edge $\{x_1, x_2\}$ uncolored, we assign to $\{x_1, x_2\}$ the color of the edge $\{\phi(x_1), \phi(x_2)\}$. Call such an embedding *good* if pulling back the embedding does not introduce any monochromatic triangles into the partial coloring that results from pulling back the embedding onto C.

All the remaining statements and propositions in this chapter assume

```
              Coloring on 29 vertices with 4 colors.
             Vertices in attaching set marked with x.

                          xxxxx

          11111222223333300000000000004
         1 2233113221133200000000000004
         12 323122133213100000000000004
         123 32212312331100000000000004
         1323 2231213112300000000000004
         13322 321121321300000000000004
         211223 31313231200000000000004
         2121323 1132313200000000000004
         23221111 333322100000000000004
         221321313 12123300000000000004
         2231121331 1232300000000000004
         31323132321 21211233122333144     x
         312313233122 2113133233221144     x
         3313123122312 123332312311244     x
         33312113232211 21332231132344     x
         321133221331122 2231313123344     x
         0000000000013312 111222233340
         00000000000213321 23113213240
         000000000003333312 2212121140
         0000000000033221132 321211340
         00000000000123232123 31131240
         000000000002313121123 1323240
         0000000000023213232111 332140
         00000000000323112212133 12340
         000000000003213231213231 2140
         0000000000031123331113222 240
         00000000000112333213221312 40
         000000000004444444444444444 0
         4444444444444444000000000000

   color  ccccccccccc00000ccccccccccccc, 144 edges in color 0
  degree  55555555555888885555555555500,  75 edges in color 1
    seq:  55555555555888885555555555500,  75 edges in color 2
          55555555555aaaaa5555555555500,  80 edges in color 3
          11111111111222221111111111gg,  32 edges in color 4
```

Figure 5.3: Partial coloring with vertex set $N_\delta(u) \cup N_\delta(v) \cup \{u, v\}$ from Υ_3.

the configuration of vertices is within a good 4-coloring of K_{62}.

Proposition 5.3: A good 4-coloring on the set on vertices $N_4(u) \cup N_4(v) \cup \{u, v\}$ within K_{62}, must contain as a partial subcoloring one of the 454 outputs obtained from colorings in Υ_3 by pulling back all good embeddings into good 3-colorings of K_{16} of the neighborhoods of the first vertex in the attaching set.

Proof: The following algorithm was implemented. For each coloring C in Υ_3 recapture the attaching set. Fix a vertex in the attaching set. Fix a color c in $\{1, 2, 3\}$. Look at the fixed color neighborhood of the fixed vertex. Embed that neighborhood into each of the two good 3-colorings of K_{16} in colors $\{1, 2, 3, 4\} \setminus \{c\}$ in all possible ways. For each such embedding check that no triangles are introduced when "pulling back" the embedding to C. If, for every vertex in the attaching set, there exists such a good embedding in two out of the three colors from $\{1, 2, 3\}$ then write to file each coloring obtained by pulling back a good embedding of the first vertex in the attaching set onto C.

That is, for each coloring in Υ_3 let V be the vertex set of the underlying coloring and let A be the vertices in the attaching set. Then $\forall\ x \in A$, $\forall\ c \in \{1, 2, 3\}$ find all good embeddings of $N_c(x)$ into $T_1(c)$ and $T_2(c)$. If $\forall\ x \in A$ there exists a good embedding in two out three colors from $\{1, 2, 3\}$ then write to file the partial coloring on V obtained by pulling back each of the good embeddings into $T_1(c)$ and $T_2(c)$ of the first vertex in A.

Pass the file through *shortmc* to eliminate isomorphic copies. 454 partial colorings resulted. $\square$

Each of these 454 results is a partial coloring of a graph with vertex set $N_4(u) \cup N_4(v) \cup \{u, v\}$, where now the first vertex in the attaching set has one of its neighborhoods fully colored in colors $1, 2, 3$. We refer to the colorings that result from this phase as *partial colorings with marked attaching sets extended by one vertex.* See Table 5.4 for a breakdown of these partial colorings by order of attaching set. Two of these partial colorings had attaching sets of order 15, which are not needed for the final result and these two partial colorings were discarded at this point. The only remaining orders for attaching sets at this point were $2, 4, 5$ and 6. Let Υ_4 denote the set of all partial colorings with marked attaching sets extended by one vertex having attaching set of order less than 15, i.e. $|\Upsilon_4| = 452$. Agreement was reached in independent computations of Υ_4 (and on the two partial colorings whose attaching sets were of order 15).

order of attaching set	number of partial colorings
15	2
6	7
5	13
4	103
2	329
total	454

Table 5.4: Partial colorings in Υ_4.

Example 5.4 : We give a sample coloring from Υ_4 in Figure 5.4. This example is the coloring from Figure 5.3 with nine additional edges colored. The x's denote a row and column in the upper triangle of the incidence matrix that has a new entry. For instance, the row 2 column 17 entry is now a 4. The first vertex in the attaching set is vertex 12. The additional colorings of the nine edges are a result of pulling back a good embedding of $N_1(v_{12})$. Note the number of uncolored edges (those assigned color 0) decreases in consecutive phases of our analysis.

We now attack the problem of coloring all the edges in the neighborhoods of the vertices in the attaching set simultaneously. For a partial coloring in Υ_4 to be contained in a good 4-coloring of a K_{62}, each vertex in the attaching set must have the property that for at least two out of the three colors 1, 2, 3, the neighborhood of the vertex in that color c must have a good embedding into $T_1(c)$ or $T_2(c)$. Now we overlap these good embeddings.

Suppose x, y are vertices in the attaching set of a partial coloring in Υ_4. For $c, d \in \{1, 2, 3\}$ suppose $N_c(x), N_d(y)$ each have a good embedding into $T_1(c)$, $T_2(c)$, and $T_1(d)$, $T_2(d)$, respectively. We say that two good embeddings *overlap* if an uncolored edge in the partial coloring that is colored by each of the pullbacks is assigned the same color by each of the good embeddings. Moreover, we say that the pullbacks overlap *successfully* if pulling back both good embeddings introduces no monochromatic triangles.

We extend the definition of successfull overlap to more than two embeddings. Let C be a partial coloring. Suppose we have a sequence $\phi_1, \phi_2, \ldots, \phi_n$ of good embeddings. So, $\forall\, i \in \{1, 2, \ldots, n\}$, $\phi_i : N_c(x) \to V(T_j(c))$ for some $c \in \{1, 2, 3\}$, x in the attaching set of C, $j \in \{1, 2\}$, is a good embedding. Define a sequence of colorings recursively by $C_1 = C$, for $i > 1$ let C_i be

```
   Colorings on 29 vertices with 4 colors

                        x   x     x

         1111122222333330000000000004
        1 22331132211332400040000020 4     x
        12 3231221332131000000000000004
        123 322123123311000000000000004
        1323 22312131123000000000000004
        13322 3211213213400020000040 4     x
        211223 31313231200000000000004
        2121323 1132313200000000000004
        23221111 333322100000000000004
        221321313 12123300000000000004
        2231121331 1232320004000004 04     x
        31323132321 21211233122333144
        312313233122 2113133233221144
        3313123122312 123332312311244
        33312113232211 21332231132344
        321133221331122 2231313123344
        0400040000213312 111222233340
        00000000000213321 23113213240
        000000000003333312 2212121140
        0000000000033221132 321211340
        04000200004123232123 31131240
        000000000002313121123 1323240
        0000000000023213232111 332140
        00000000000323112212133 12340
        000000000003213231213231 2140
        0000000000031123331113222 240
        02000400004112333213221312 40
        000000000004444444444444444 0
        4444444444444444000000000000

 color  c9ccc9cccc9000009ccc9ccccc9cc, 135 edges in color 0
degree  55555555555888885555555555500,  75 edges in color 1
   seq: 56555655556888886555655555600,  78 edges in color 2
        55555555555aaaaa5555555555500,  80 edges in color 3
        13111311113222223111311111 3gg,  38 edges in color 4
```

Figure 5.4: Marked attaching set extended by one vertex from Υ_4.

the coloring obtained by pulling back ϕ_i onto C_{i-1}. We say $\phi_1, \phi_2, \ldots, \phi_n$ *overlap successfully* if at each step C_i contains no monochromatic triangles and refer to such a sequence as a *good* sequence. We call C_n the *pullback* of the sequence of embeddings.

Proposition 5.4: A good 4-coloring on the set of vertices $N_4(u) \cup N_4(v) \cup \{u, v\}$ within K_{62}, where the attaching set has order less than 15, must contain as a partial subcoloring one of the 512 outputs (5 with attaching set of order 5 and 507 with attaching set of order 2) obtained from colorings in Υ_4 by pulling back all possible good sequences of embeddings obtained by using two out of three of the colors $1, 2, 3$ for each vertex in the attaching set.

Proof: The following algorithm was implemented. For each partial coloring in Υ_4 of a marked attaching set extended by one vertex, for each vertex in the attaching set, for each color neighborhood, embed this neighborhood into T_1 and T_2 in the other three colors, in all ways and for each good embedding hold the information. In all possible ways for two out of three colors overlap the embeddings (in the sense that overlapping edges must agree in color) so that no monochromatic triangles result.

That is, for each coloring in Υ_4 let V be the vertex set of the underlying graph and let $A = \{x_1, x_2, \ldots, x_n\}$ be the vertices in the attaching set. Then $\forall\ x\ \in A$, $\forall\ c\ \in \{1, 2, 3\}$ find all good embeddings of $N_c(x)$ into $T_1(c)$ and $T_2(c)$. If $\forall\ x \in A$ there exists a good embedding in two out three colors from $\{1, 2, 3\}$ then continue.

In all possible ways, successfully overlap the pullbacks of good embeddings $\forall\ x\ \in A$, for two out of three colors from $\{1, 2, 3\}$. That is, find all possible sequences of embeddings $\phi_{1,c_1},\ \phi_{1,d_1},\ \phi_{2,c_2},\ \phi_{2,d_2}, \cdots, \phi_{n,c_n},\ \phi_{n,d_n}$ where each $(c_i, d_i) \in \{(1, 2), (1, 3), (2, 3)\}$ and for $b_i \in \{c_i, d_i\}$, $\phi_{i,b_i} : N_{b_i}(x_i) \rightarrow V(T_k(b_i))$, for some $k\ \in\ \{1, 2\}$ that overlap successfully. For each such good sequence, write to file the pullback of the sequence.

Pass the file through *shortmc* to eliminate isomorphic copies. 512 partial colorings resulted. □

Each of these 512 results is a partial coloring with vertex set $N_4(u) \cup N_4(v) \cup \{u, v\}$ obtained by coloring additional edges in a partial coloring $C\ \in\ \Upsilon_4$. Now each vertex in the attaching set has two out of its three neighborhoods from C in colors $1, 2, 3$ fully colored. The only orders for $N_4(u) \cap N_4(v)$ left at this point are 2 and 5. There are 507 partial colorings with attaching set having order 2 and 5 partial colorings with attaching set having order 5. Since the partial colorings with attaching set having order

2 were not needed for our result, we discarded them. Let Υ_5 denote the set of all (5) partial colorings with attaching set of order 5 that result from implementing the algorithm from Proposition 5.4. Agreement was reached in independent computations of Υ_5 (as well as on the 507 partial colorings with attaching set having order 2).

Example 5.5 : We give a sample coloring from Υ_5 in Figure 5.5. This example is the coloring from Figure 5.4 with 61 additional edges colored as a result of successfully overlapping the pullbacks of good embeddings of $N_c(x)$ for two out of three $c \in \{1,2,3\}$, $\forall\ x = v_k,\ k \in \{12,13,14,15,16\}$.

We now color additional edges by implementing a somewhat similar algorithm for vertices in $S = (N_4(u) \cup N_4(v) \cup \{u,v\}) \backslash (N_4(u) \cap N_4(v))$, and for all four colors. Let $x \in S$ be a vertex in a partial coloring from Υ_5. For $c \in \{1,2,3,4\}$, we say c is a *feasible* color for x if $N_c(x)$ is fully colored or if it is not fully colored and $N_c(x)$ has a good embedding in $T_i(c)$, for some $i \in \{1,2\}$.

Proposition 5.5: A good 4-coloring on the set of vertices $N_4(u) \cup N_4(v) \cup \{u,v\}$ within K_{62}, where the attaching set has order 5, must contain as a partial subcoloring one of the 8191 colorings obtained from colorings in Υ_5 by successfully overlapping in all possible ways the pullbacks for three out of four of the colors from $\{1,2,3,4\}$ of the good embeddings of the neighborhood in that color c into $T_i(c), i = 1,2$ for each vertex not in the attaching set that has exactly three feasible colors.

Proof: The following algorithm was implemented. For each coloring $C \in \Upsilon_5$ let V be the vertex set of the underlying coloring, let A be the set of vertices in the attaching set and let $S = V \backslash A$. For each $x \in S$, for each $c \in \{1,2,3,4\}$ find all good embeddings of $N_c(x)$ into $T_1(c)$ and $T_2(c)$. Count feasible colors for x. Let $f(x)$ denote the number of feasible colors for x. The actions depending on $f(x)$ are: If $f(x) < 3$ then reject the input coloring and quit. If $f(x) = 3$ then store the good embeddings for later use. If $f(x) = 4$ then ignore this vertex, and continue. $\forall\ x$ such that $f(x) = 3$, for the three feasible colors c for x, in all possible ways successfully overlap the pullbacks of the good embeddings into $T_i(c), i = 1,2$. Write the result to file. Each such resulting partial coloring has had additional edges colored.

After passing the output through *shortmc,* 8191 partial colorings resulted. $\square$

```
         Colorings on 29 vertices with 4 colors

              1111122222333330000000000004
             1 2233113221133224201440004004
             12 323122133213110424000042404
             123 32212312331104044200041204
             1323 2231213112342400004024104
             13322 321121321341042420000404
             211223 31313231240024440110004
             2121323 1132313201244010440004
             23221111 333322114440100204004
             221321313 12123304410420041040
             2231121331 1232340101240404004
             31323132321 21211232311332344
             312313233122 2112113231323344
             3313123122312 123213323311244
             33312113232211 22321113333244
             321133221331122 3331232312144
             0210444010412323 111122233340
             04042101440212331 23312211340
             024040024413112312 3223123140
             0024042441023311133 222131240
             01440244001323121322 31213140
             040204401421321321223 3132140
             0400424102411332223213 121340
             00000000000333332211211 12240
             004420142043213131231321 2240
             0421401404023132313132122 140
             00421400414332213312113221 40
             000000000004444444444444444 0
             4444444444444444000000000000

   color  c555555555500000555555c555cc,  74 edges in color 0
  degree  56666677777888887766766567700,  90 edges in color 1
    seq:  57777766666888886677677576600,  90 edges in color 2
          55555555555aaaaa5555555555500,  80 edges in color 3
          15555555555222225555551555gg,  72 edges in color 4
```

Figure 5.5: A coloring in Υ_5.

Each of these 8191 results is a partial coloring of a graph with vertex set $N_4(u) \cup N_4(v) \cup \{u, v\}$ obtained by coloring additional edges in a partial coloring $C \in \Upsilon_5$. Now each vertex not in the attaching set that had exactly three feasible colors has all three of those feasible color neighborhoods from C fully-colored. This, of course, changes some of the neighborhoods so that the new (possibly larger) neighborhoods need not be fully colored. Let Υ_6 denote the set of all partial colorings that result from implementing the algorithm from Proposition 5.5. Agreement was reached in independent computations of Υ_6.

Note that we could have mimicked the proof of Proposition 5.4 at this stage instead. That is, for vertices in S, we could have found all sequences of embeddings using three out of four colors from {1,2,3,4} for each vertex, that overlap successfully and write to file the pullback of each such sequence. In fact such a program was written. However, considering colorings obtained by the pullback of sequences of embeddings using the exact three colors that were feasible and for only those vertices in S with exactly three feasible colors sufficed, and it was Υ_6 that was compared.

Example 5.6 : We give a sample coloring from Υ_6 in Figure 5.6. This example is the coloring from Figure 5.5 with 47 additional edges colored as a result of successfully overlapping the pullbacks of good embeddings of $N_c(x)$ $\forall\ x = v_k,\ k \in \{1, 2, 3, \ldots, 11, 17, 18, \ldots, 29\}$ such that v_k has exactly three feasible colors for each feasible color c.

The final phase consisted of running each of the independent programs used in Proposition 5.5 to obtain Υ_6 but now with Υ_6 as input. No colorings were obtained by either program. Thus we have shown:

Theorem 5.6: There does not exist a good 4-coloring of K_{62}.

Proof: In Theorem 4.2 we showed that every good 4-coloring of K_{62} contains an attaching set of order k, where $3 \leq k \leq 14$. Propositions $5.1 - 5.5$ above along with the final phase that obtained no output show there is no good 4-coloring for such an attaching set.$\square$

```
Colorings on 29 vertices with 4 colors.

          1111122222333332244211012204
         1 223311322113322420144104024
         12 32312213321311042430442414
         123 3221231233110404420141214
         1323 223121311234243004124134
         13322 32112132134114242430434
         211223 3131323124302444411024
         2121323 113231320124401044024
         23221111 33332211444010220414
         221321313 1212334441042214134
         2231121331 123234010124244434
         31323132321 21211232311332344
         312313233122 2112113231323344
         3313123122312 123213323311244
         33312113232211 22321113333244
         321133221331122 3331232312144
         2210444014412323 111122233341
         24042131440212331 23312211342
         424041024413112312 3223123143
         4024342441023311133 222131242
         21440244001323121322 31213143
         143204401421321321223 3132141
         1400424102411332223213 121341
         01411440222333332211211 12240
         104423142143213131231321 2243
         2421401404423132313132122 142
         20421400414332213312113221 43
         021133221334444444444444444 0
         4444444444444444123231103230

   color  2323212431200000222323312322,  27 edges in color 0
  degree  8778777788788888877678888773 3, 104 edges in color 1
    seq: a877777777888887878777878733, 104 edges in color 2
         55657765566aaaaa5666665575644,  92 edges in color 3
         3565566556622222656655545 65gg,  79 edges in color 4
```

Figure 5.6: A coloring in Υ_6.

Chapter 6

Software Tools

We have at our disposal the software to store and manipulate multicolored graphs in the .mc format developed by S. Radziszowski and used in [18]. The .mc format allows each graph coloring to be represented by one line with two security bits in each byte. That is, given a *c-coloring* of a graph G on n vertices, the .mc encoding of G consists of a string of characters formed by one byte for n (six bits, two padded), one byte for c (six bits, two padded), followed by $n(n-1)/2$ blocks of k bits where $k = \lceil (log_2(c+1)) \rceil$, broken into six-bit pieces.

Example 6.1: Consider the good 2-coloring of K_5 given in Theorem 3.1, identifying colors α, β with 1 and 2, respectively. The incidence matrix for this coloring is:

```
 1221
1 122
21 12
221 1
1221
```

Here $n = 5$, $c = 2$ and $k = 2$, so we should have one byte for 5, one byte for 2 followed by ten blocks of two bits broken into six bit pieces. To obtain this representation, first write the upper triangular portion of the matrix as a sequence by proceeding left to right down each column, and preceded by 5 and 2. We have:

5|2|121|221|122|100

which becomes (in binary):

101|10|011001|101001|011010|010000.

Pad the string with 0's (on the left) to 8-bit bytes::

00000101|00000010|00011001|00101001|00011010|00010000.

Now write each byte in its usual decimal representation and add to each +63 in order to have printable characters:

68|65|88|104|89|79.

Use the keys from the ASCII table and we see the string for the incidence matrix is:

DAXhYO. □

Nauty, a program that computes a canonical labeling of graphs, was developed by B. McKay [15]. For a graph G, a *canonical labeling,* $can(G)$, of $V(G)$, is a labeling with the property that two graphs, G_1, G_2, are isomorphic if and only if $can(G_1) = can(G_2)$. Thus, we translate the isomorphism problem into identity which is then solved by the standard UNIX sort -u, utility which deletes identical lines.

The interface between the .mc format and *Nauty,* called *shortmc* was also developed by B. McKay [15]. *Shortmc* run on an input file of colorings held in the .mc format results in a file containing a subset of the original colorings, one from each isomorphism class, each with the canonical labeling. The -k option on *shortmc* gives a labeling from one of the representatives of each isomorphism class. We utilized this option so the vertices in the attaching set retained the same labels after processing the ouput files.

Chapter 7

Conclusions

As mentioned in Chapter 3, R. Kramer's manuscript [13] was the main motivation for beginning this thesis. So, we include here some comments about his work. Kramer's extensive local arguments contained a section for each of the possible orders for potential attaching sets. With a single exception, that of order 5, he was able to prohibit the existence of the attaching set in question. Our work supports the nonexistence of all such potential attaching sets and we note that those of order 5 were the only ones left after Proposition 5.4. Further examination of the structure of those partial colorings should show that they satisfy the conditions set forth by Kramer for potential attaching sets of order 5. Kramer had the most difficulty (requiring 49 pages) showing the nonexistence of attaching sets of order 4, an order that for us, dropped out with order 6 at the Proposition 5.4 stage.

Can we push our approach further? Suppose K_n has a good coloring C in four colors. For $n \geq 60$ for at least three out of four colors, the orders of the neighborhoods must be at least 14 and for at least two the orders are at least 15. All $R(3,3,3;14)$ colorings are known [17]. Further, our proof of Theorem 4.2 can be modified to show the existence of a $u-v$ attaching set $N_\delta(u) \cap N_\delta(v)$ in C of order k, where $3 \leq k \leq 13$, with $|N_\delta(u)| = |N_\delta(v)| \geq 15$. Note that Propositions 5.1 and 5.2 give marked subsets of the two (3,3,3;15) colorings and ways to overlap two good 3-colorings of K_{15}, respectively, and these results were in agreement having been reached by two independent computations. It was at the Proposition 5.3 stage the the number of partial colorings with marked attaching sets extended by one vertex became too numerous to handle. Perhaps better file management and more computing power might yield results.

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